THE TWENTY-FOURTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

At the invitation of Adelbert College and the Case School of Applied Science, Cleveland, Ohio, the twenty-fourth summer meeting of the Society was held at these institutions on Tuesday, Wednesday, and Thursday, September 4–6, 1917. This was the Society's second visit to Cleveland, the annual meeting having been held there in the winter of 1912–1913. On the present occasion the interest was reinforced by the meeting of the Mathematical Association of America, immediately following on September 6–7. The arrangements, which were in charge of a committee representing both organizations, included a joint session on Thursday morning, at which Professor L. P. Eisenhart presented an address on "Darboux's Contribution to Geometry," and a joint dinner on Wednesday evening, attended by seventy-six members and friends, to whom President Thwing, of Western Reserve University, spoke a word of greeting, which was followed by a number of informal responses to the calls of the toast-master, Professor Huntington. The programme on Wednesday afternoon included an inspection of the harmonic analysis apparatus of Professor Miller, of the Case School, and an organ recital in the chapel. On Thursday afternoon President Thwing gave a garden party in honor of the visiting societies. Luncheon was served on each day at the Case Club, whose building was thrown open to the members afternoons and evenings. At the close of the meeting a vote of thanks was tendered to the authorities of the two colleges for their generous hospitality.

The meeting included the usual morning and afternoon sessions on Tuesday and Wednesday and the joint session on Thursday morning. The following sixty-four members were in attendance:

Professor O. P. Akers, Dr. Florence E. Allen, Professor R. B. Allen, Professor L. D. Ames, Professor Frederick Anderegg, Professor R. C. Archibald, Professor G. N. Armstrong, Professor Grace M. Bareis, Professor J. W. Bradshaw, Dr. R. W. Burgess, Professor W. D. Cairns, Professor Florian Cajori, Professor W. M. Carruth, Professor W. B. Carver, Mr. E. H.

Clarke, Dr. G. R. Clements, Professor F. N. Cole, Professor A. R. Crathorne, Professor C. H. Currier, Professor F. F. Decker, Professor L. L. Dines, Professor L. W. Dowling, Professor John Eiesland, Professor L. P. Eisenhart, Professor T. M. Focke, Professor Tomlinson Fort, Dr. M. G. Gaba, Professor D. C. Gillespie, Professor O. E. Glenn, Professor W. C. Graustein, Professor C. F. Gummer, Professor A. M. Harding, Professor H. E. Hawkes, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor F. H. Hodge, Professor E. V. Huntington, Professor W. A. Hurwitz, Dr. R. A. Johnson, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor H. G. Keppel, Professor G. A. Miller, Professor W. L. Miser, Professor C. N. Moore, Professor F. R. Moulton, Professor A. D. Pitcher, Professor L. C. Plant, Professor S. E. Rasor, Professor R. G. D. Richardson, Professor H. L. Rietz, Professor Maria M. Roberts, Professor E. D. Roe, Jr., Professor D. A. Rothrock, Professor J. E. Rowe, Professor C. H. Sisam, Professor P. F. Smith, Professor Virgil Snyder, Professor R. P. Stephens, Professor E. B. Stouffer, Professor M. O. Tripp, Professor D. T. Wilson, Professor B. F. Yanney, Professor J. W. Young.

At the opening session on Tuesday morning Professor Focke occupied the chair, which was filled in succession by Professors Hedrick, Cajori, G. A. Miller, and Eisenhart. Professor Hedrick presided at the joint session. The Council announced the election of the following persons to membership in the Society: Dr. W. L. Crum, Yale University; Professor T. J. Fitzpatrick, University of Nebraska; Dr. T. R. Hollcroft, Columbia University; E. L. Ince, M. A., Trinity College, Cambridge, England; Mr. L. S. Odell, Manual Training High School, Brooklyn, N. Y.; Dr. T. A. Pierce, Harvard University. Five applications for membership in the Society were received.

The following papers were read at this meeting:

(1) Professor ARNOLD EMCH: "On the invariant net of cubics in the Steinerian transformation."

(2) Professor J. E. ROWE: "Theorems related to a point projection of the rational plane cubic curve."

(3) Professor J. E. ROWE: "Closed hexagons related to the rational plane cubic curve."

(4) Professor J. E. ROWE: "The projections of certain points upon the rational plane quartic curve."

(5) Professor TOMLINSON FORT: "Some theorems of comparison and oscillation."

(6) Professor O. D. KELLOGG: "Oscillation and interpolation properties of solutions of integral equations."

(7) Professor A. B. COBLE: "Finite groups determined by $2p + 2$ points in S_p."

(8) Dr. M. G GABA: "Complete existential theory of the postulates of the linear order η."

(9) Professor L. L. DINES: "The bordered Fredholm determinant and the related group of functional transformations."

(10) Professor R. G. D. RICHARDSON: "Contributions to the study of oscillation properties of ordinary linear differential equations of the second order."

(11) Professor C. N. MOORE: "On the summability of the developments in Bessel's functions."

(12) Professor G. A. MILLER: "Groups formed by special matrices."

(13) Professors VIRGIL SNYDER and F. R. SHARPE: "On the space involution of order 8 defined by a web of quadric surfaces."

(14) Dr. R. W. BURGESS: "A second approximation for cantilevers."

(15) Professor FLORIAN CAJORI: "L. Wantzel."

(16) Dr. G. M. GREEN: "Conjugate nets with equal point invariants."

(17) Dr. G. M. GREEN: "Plane nets with equal invariants."

(18) Professor FLORIAN CAJORI: "Newton's solution of numerical equations by the use of slide rules."

(19) Professor L. P. EISENHART: "Transformations of planar nets with equal invariants."

(20) Dr. L. C. MATHEWSON: "On the group of isomorphisms of a certain extension of an abelian group."

(21) Professor E. D. ROE, JR.: "Some restricted developments."

(22) Professor E. D. ROE, JR.: "A geometric representation. Second paper."

(23) Professor E. D. ROE, JR.: "Integral functions as products."

(24) Mrs. E. D. ROE, JR.: "Interfunctional expressibility problems of symmetric functions. Second paper."

(25) Professor E. L. DODD: "The approximation or graduation of a mortality table by means of a sum of exponential functions."

(26) Professor D. C. GILLESPIE: "Repeated integrals."

(27) Professor W. A. HURWITZ: "An expansion theorem for systems of linear differential equations."

(28) Professor W. C. GRAUSTEIN: "Note on isogenous complex functions of curves."

(29) Dr. MARY F. CURTIS: "A proof of the existence of the functions of the elliptic cylinder."

(30) Professor JOHN EIESLAND: "A Plücker geometry of flats in odd n-space."

(31) Professor H. J. ETTLINGER: "Theorems of oscillation for a generalized Sturmian boundary problem."

(32) Professor H. J. ETTLINGER: "Theorems of oscillation for the general real, self-adjoint system of the second order."

(33) Professor E. V. HUNTINGTON: "Bibliographical note on the use of the word mass in current textbooks."

(34) Professor L. P. EISENHART: "Darboux's contribution to geometry."

Dr. Curtis was introduced by Professor Bôcher. Dr. Green's second paper was presented by Professor Eisenhart, and Mrs. Roe's paper by Professor Roe. The papers of Professor Emch, Professor Coble, Dr. Mathewson, Professor Dodd, Professor Eiesland, and Professor Ettlinger, and Dr. Green's first paper were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond with the titles in the list above.

1. Considering $A_1(1, 0, 0)$; $A_2(0, 1, 0)$; $A_3(0, 0, 1)$; $E(1, 1, 1)$ as the fundamental triangle and unit point of a system of projective coordinates, the formulas for the Steinerian transformation associated with the quadrangle $A_1A_2A_3E$ are

$$\rho x_1' = x_1(x_2 + x_3 - x_1),$$

$$\rho x_2' = x_2(x_3 + x_1 - x_2),$$

$$\rho x_3' = x_3(x_1 + x_2 - x_3).$$

The net of cubics invariant under this transformation is represented by

$$a_1x_2x_3(x_2 - x_3) + a_2x_3x_1(x_3 - x_1) + a_3x_1x_2(x_1 - x_2) = 0.$$

Every triad of values $a_1 : a_2 : a_3$ defines a cubic of the net through the quadrangle and its diagonal points. Every cubic has the interesting property that the tangents at A_1, A_2, A_3, E meet in a point of the cubic with the coordinates a_1, a_2, a_3. Professor Emch discusses some geometric properties of this net of cubics.

2. In Professor Rowe's first paper four theorems are established, the first two of which are summarized in the statement: If $\varphi_{t_1}{}^2\varphi_{t_2}{}^2 = 0$ is the equation of the line determined by the two points t_1, t_2 of the rational plane cubic curve R^3, the discriminant of this quadratic in t_2, equated to zero, yields a quartic in t_1 whose roots are the parameters of the residual points cut out of the R^3 by the four tangents to the R^3 from the point x_i $(i = 0, 1, 2)$ of the plane; or for a fixed value of t_1 this same equation is the equation of a tritangent conic of the R^3 whose points of contact are t_1 and the points of contact of the two tangents that may be drawn from t_1 to the R^3. Theorems III and IV follow.

III. The parameters of the points of contact of the two tangents to the R^3 from any point of itself are harmonic to the parameters of the node.

IV. If t_2, t_3 are the parameters of the points of contact of the two tangents to the R^3 from the point t_1, and t_1' is the third point of the R^3 collinear with t_2, t_3, the transformation which sends t_1 into t_1' is involutory.

3. In this paper Professor Rowe derives one form of the necessary and sufficient condition that six points of the rational plane curve R^3 of order three lie on a conic, and uses this condition to prove the following theorems:

I. If six points of the R^3 lie on a conic, the tangents to the R^3 at these six points cut out of the R^3 six points on a conic.

II. If six points of the R^3 lie on a conic, the sides of the closed hexagon formed with these points as vertices cut the R^3 in six points on a conic.

III. If from a point of the R^3 the two tangents are drawn to the R^3 and from each point of contact of these two tangents the other two tangents to the R^3 are drawn, the six points of contact of the six tangents lie on a conic.

By a limit process it is evident that Theorem I may be derived as a corollary of Theorem II, although a different proof is given for each.

Also, it may be verified that the six points of contact of the three pairs of tangents drawn to the R^3 from three of its collinear points do not, in general, lie on a conic.

4. In this paper Professor Rowe proves the following four theorems together with three corollaries:

I. The parameters of the points of contact of the two tangents that can be drawn to the rational plane quartic curve R^4 from one of its nodes are harmonic to the parameters of the residual points cut out of the R^4 by any line through this node.

II, III, IV are summarized in the statement: If t_1, t_2 are the parameters of the points of contact of two flex tangents, of two double tangents, or of a flex tangent and a double tangent of the R^4, which cut out of the R^4 the residual points t_1', t_2', and if t_1'', t_2'' are the parameters of the points of contact of the other two tangents that can be drawn to the R^4 from the intersection of the tangents at t_1, t_2, the pairs (t_1, t_2), (t_1', t_2'), (t_1'', t_2'') form a set in involution.

5. Let y denote a solution of the differential equation $d(k(x)y')/dx + G(x)y = 0$. Professor Fort develops a number of theorems relative to $\alpha_1 k(x)y' + \alpha_2 y \equiv V(x)$ which reduce to well known theorems of comparison, oscillation, etc., in case that $V(x) \equiv y$.

6. In the *American Journal of Mathematics* for January, 1916, Professor Kellogg showed that continuity and orthogonality of a function set on the interval (0, 1) was insufficient to insure oscillation, but that if to these assumptions is added the hypothesis (D): the determinants $|\varphi_i(x_j)|$ are positive for $0 < x_0 < x_1 < x_2 \cdots < 1$, a considerable number of the properties of the more common sets of orthogonal functions follow. The present paper derives a sufficient condition on the kernel of an integral equation in order that its characteristic solutions may have the property (D) and its consequences.

7. It is the purpose of Professor Coble's paper (to appear in the *American Journal of Mathematics*) to show that a set

of $2p+2$ points in a projective space S_p, which is such that the quadrics on $2p+1$ of the points will pass through the remaining one, will define by purely projective processes a finite group which is isomorphic with the group determined by the odd and even theta functions in p variables. Since both the number of absolute constants of the point set and the number of moduli of the general theta function is $\frac{1}{2}p(p+1)$, this indicates an algebraic connection between such point sets and the general theta modular functions. This connection has been set forth hitherto only in the particular cases, $p=1, 2, 3$, and in the hyperelliptic case for all values of p; although in an earlier paper of the writer a grouping of the theta characteristics isomorphic with a grouping of the points of the set has been exhibited. It appears thereby that the point set may serve as the basis for a geometrical classification of the general theta functions, just as the algebraic curve of genus p has served for the discussion of the Riemannian theta functions of genus p.

8. In the March, 1917, BULLETIN, Professor E. V. Huntington gave three sets of completely independent postulates for serial order. His set A involved four postulates, which is as high a number as has been proved to be completely independent. In the present paper Dr. Gaba states seven postulates for the linear order η which form a categorical and completely independent set. For his 128 independence examples he gives 16 sets of points and 8 definitions of $<$ such that each definition is applicable to every one of the 16 sets of points and each combination of set and definition of $<$ yields a different example.

9. The determinant of the projective transformation

$$\phi'(x) = \frac{\alpha(x) + \beta(x)\phi(x) + \int_0^1 \gamma(x, y)\phi(y)dy}{\delta + \int_0^1 \epsilon(y)\phi(y)dy},$$

is defined by Professor Dines to be the bordered Fredholm determinant

$$B = D\left[\delta - \int_0^1 \epsilon(y)\bar{\alpha}(y)dy\right] + \int_0^1\int_0^1 \epsilon(x)D_1(x, y)\bar{\alpha}(y)dxdy,$$

where $\bar{\alpha}(x) = \alpha(x)/\beta(x)$, and D and $D_1(x, y)$ are respectively the Fredholm determinant and first minor of the kernel $\gamma(x, y)/\beta(x)$. The following results are obtained:

(1) The determinant of the product of two projective transformations is equal to the product of their determinants.

(2) If the determinant B of a projective transformation is different from zero, a unique inverse transformation exists and is expressible in terms of B and four suitably defined first minors of B.

(3) The group of non-singular projective transformations (those for which $B \neq 0$) contains every finite transformation generated by a regular infinitesimal projective transformation as defined by Kowalewski; and, conversely, any non-singular projective transformation which does not differ too greatly from the identity transformation can be generated by such an infinitesimal transformation.

10. The Sturmian problem of determining the existence of parameter values λ for which the equation

$$(py_x)_x + G(x, \lambda)y = 0, \qquad p(x) > 0,$$

has solutions $y(x)$ satisfying linear self-adjoint boundary conditions has been much studied on account of its importance in mathematical physics and because it is a typical linear problem. Professor Richardson announces the results of further investigations as follows: (1) the derivation of sufficient conditions to be imposed on the function $G(x, \lambda)$ to insure the existence of solutions oscillating as many times as desired; (2) the determination of conditions on $G(x, \lambda)$ such that for a given interval of λ there is precisely one or precisely two parameter values for which exist solutions satisfying the boundary conditions and oscillating a prescribed number of times; (3) the completion of the theory of oscillation for the most important special case $G(x, \lambda) \equiv q(x) + \lambda k(x)$, viz., an investigation of the only case not hitherto studied in detail. For the non-definite case ($k(x)$ both signs, $q(x)$ positive in at least a part of the x-interval) it is found that there exist two integers ν_1, ν_2 ($\nu_2 \geqq \nu_1 \geqq 0$) such that if n denote the number of oscillations of the solution then for $n < \nu_1$ there are no solutions, for $n > \nu_1$ there are at least two, and for $n \geqq \nu_2$ there are precisely two.

11. The principal theorems obtained in Professor Moore's paper are the following: (A) If $f(x)$ is integrable (Lebesgue) in the interval $0 \leqq x \leqq 1$, approaches a finite limit as x approaches $+0$, and furthermore is such that $(f(x) - f(+0))/x^{\delta}$, where $\delta > \frac{1}{2}$, remains finite in the neighborhood of the origin, then the development of $f(x)$ in Bessel's functions of order $\nu(\nu \geqq 0)$ will be summable $(C, k > \frac{1}{2})$ to the value $f(+0)$ for $x = 0$, provided $\nu = 0$ or $f(+0) = 0$, and will be uniformly summable $(C, k > \frac{1}{2})$ to the value $f(x)$ in any interval $0 \leqq x \leqq c \leqq 1$, throughout which $f(x)$ is continuous, provided $\nu = 0$ or $f(+0) = 0$. (B) If $\sqrt{x}f(x)$ is integrable (Lebesgue) in the interval $0 \leqq x \leqq 1$, then the development of $f(x)$ in terms of Bessel's functions of order $\nu(\nu \geqq 0)$ will be summable $(C, k > 0)$ to the value $\frac{1}{2}[f(x+0) + f(x-0)]$ at every point in the interval $0 < x \leqq 1$ at which $f(x)$ is continuous or has a finite jump, and will be uniformly summable $(C, k > 0)$ to $f(x)$ throughout any closed interval lying entirely in the interval $0 < x \leqq 1$ and throughout which $f(x)$ is continuous.

12. Any set of square matrices of order n having one and only one unity element in each row and in each column, while all their other elements are zeros, generates a finite group which is simply isomorphic with some substitution group of degree n. When these unity elements are replaced by kth roots of unity the resulting matrices generate a group which is simply isomorphic with a substitution group of degree kn. When this substitution group is transitive it has n systems of imprimitivity, each system being of degree k. The main object of Professor Miller's paper is to consider the nature of these elements in order that any imprimitive group of degree kn having n systems of imprimitivity can be represented by such square matrices of order n. The following theorems are established: Every possible imprimitive group of degree $2n$ which has n systems of imprimitivity is simply isomorphic with a group formed by square matrices of order n having one and only one ± 1 element in each row and in each column while all their other elements are zeros. If an imprimitive group of degree kn has for its head H the direct product of n regular cyclic groups of order k, then every invariant imprimitive subgroup has for its head a subgroup of H which is simply isomorphic with the direct product of $n - 1$ cyclic groups of order k and of some subgroup of such a cyclic group.

13. The involution discussed by Professors Snyder and Sharpe is defined by associating the planes of one space with the quadric surfaces of a web in another. We have a (1, 8) correspondence between the points of the two spaces, and an involution of order 8 in the space of the quadrics. The surface of coincident points, the surface of branch points, and the residual surface are obtained and described. The dual of the surface of branch points is the symmetroid, the characteristic properties of which are easily obtained by this method. As particular cases we meet with the Weddle surface, the Kummer surface, the Hessian of the cubic, and all of the line congruences of order 2.

14. The engineer uses constantly certain simple formulas for the deflection of a horizontal beam fixed at one end and bent by a vertical force at the other, and for the angle of slope of the free end. The treatment by which these formulas are derived can not readily be modified to give true results for large deflections; but by using the equation for the real elastic curve and developing in series the elliptic integrals involved, Dr. Burgess obtains the following formulas for the deflection d, the horizontal component t of the displacement of the free end, and the angle of slope ψ:

$$d = \frac{l^3W}{3EI} - \frac{4}{105}\frac{l^7W^3}{E^3I^3} + .008\frac{l^{11}W^5}{E^5I^5}$$

$$t = \frac{1}{15}\frac{l^5W^2}{E^2I^2} - \frac{4}{315}\frac{l^9W^4}{E^4I^4}$$

$$\tan\psi = \frac{1}{2}\frac{l^2W}{EI} - \frac{1}{240}\frac{l^6W^3}{E^3I^3} + \frac{l^{10}W^5}{600E^5I^5},$$

where E is the constant of elasticity, W the deflecting force, l the length of the beam, and I the geometric moment of inertia of a cross-section about an axis perpendicular to the plane of bending. The first terms in the expressions for d and $\tan\psi$ are the engineers' formulas; the other terms give corrections important for larger deflections, and apparently sufficient within the elastic limit.

15. Professor Cajori gave some details on the life and work of Pierre-Laurent Wantzel (1814–1848) of the Polytechnic School in Paris.

16. In his first paper, Dr. Green gives a new geometric characterization of conjugate nets with equal point invariants, which is a refinement of a theorem established by him in a previous paper.* The following theorem is first proved geometrically. Let t_1 and t_2 be the ray tangents at a point of a surface S referred to a conjugate net. On the corresponding ray there are two focal points F_1 and F_2, corresponding respectively to the ray curves of which t_1 and t_2 are the tangents. Then the tangent to S which is conjugate to t_1 meets the ray in F_2, and the tangent conjugate to t_2 meets the ray in F_1.

It is then proved that a necessary and sufficient condition that a conjugate net have equal point invariants is that the ray tangents meet the ray in the focal points of the ray, each ray tangent passing through the focal point which does not correspond to it.

If the ray tangents meet the ray in the corresponding focal points of the ray, then the ray curves coincide with the asymptotic net, and the conjugate net has not equal invariants.

17. The geometric characterization of conjugate nets with equal point invariants given in Dr. Green's first paper does not admit of immediate extension to the case of planar nets with equal invariants. For any planar net N, the ray of a point may be defined as the line joining the minus first and first Laplace transforms of the point, but every one-parameter family of rays forms a developable and gives rise to a focal point on each ray. The first object of Dr. Green's second paper is to define a net N' congruentially associated with the net N. Let t_1, t_2 be the tangents to the curves of N at a point of the plane, and let t_1', t_2' be the tangents to the curves of N' at the same point. The curve of N' to which t_1' is tangent gives rise to a focal point F_1 on the ray of the net N. Similarly, a focal point F_2 on this ray corresponds to the tangent t_2'. The net N' is said to be congruentially associated with N if the harmonic conjugate of t_1' with respect to t_1, t_2 passes through F_2, and the harmonic conjugate of t_2' with respect to t_1, t_2 passes through F_1. It is proved that there always exists one and only one net congruentially associated with the given one.

A necessary and sufficient condition that the net N have

* *Amer. Jour. of Mathematics*, vol. 38 (1916), p. 313.

equal invariants is that at every point the tangents to the two curves of N separate harmonically the tangents to the two curves of the congruentially associated net N', or, in other words, that each ray of N be met by the tangents of N' in its focal points with respect to N'.

18. Professor Cajori notes three accounts of Newton's solution of numerical equations by the use of logarithmic slide rules, only one of which is given in the collected works of Newton and is referred to in the Encyclopédie des Sciences mathématiques, Tome I, Volume 4, page 429, namely, four sentences in a letter of Oldenburg to Leibniz of June 24, 1675. A second fuller account is described in E. Stone's New Mathematical Dictionary, second edition, London, 1743 and is reprinted in Cajori's History of the Slide Rule, 1909. A third account, purporting to be taken directly from Newtonian manuscripts, is printed in James Wilson's Mathematical Tracts of the late Benjamin Robins, London, 1761. The three accounts contain four slightly different modes of procedure.

19. In the January number of the *Transactions* Professor Eisenhart developed a theory of transformations of a conjugate system of curves on any surface into conjugate systems on other surfaces. These results can be extended so as to give transformations of conjugate nets on two-dimensional spreads in space of any order. In the present paper he applies them to nets in the plane, and in particular to nets with equal point invariants. In the June number of the *Annals of Mathematics* it was shown how by quadratures one can obtain from such a net a system of asymptotic curves on a surface which are perspective with the net from a point. Two nets with equal invariants in the relation of a transformation here discussed are found to be perspective with two surfaces which are the focal surfaces of a W-congruence. The planar nets of period three under transformations of Laplace have equal invariants. The paper deals also with transformations of these nets into nets of the same kind, and establishes the existence of a theorem of permutability of these transformations. The analysis involved in this investigation is capable also of interpretation as giving transformations of quadrics, certain ruled surfaces, and surfaces of the kind first discussed by Tzitzeica into surfaces of these respective types.

20. In 1908 Professor G. A. Miller showed that "if an abelian group H which involves operators whose orders exceed 2 is extended by means of an operator of order 2 which transforms each operator of H into its inverse, then the group of isomorphisms of this extended group is the holomorph of H."* The present paper by Dr. Mathewson discusses an elaboration of Professor Miller's theorem. In toto it is proved that if G is formed by extending an abelian group H which has operators of order > 2 by a certain operator which transforms every one of its operators into the same power of itself and which is commutative with no operator of odd order in H, then the group of isomorphisms of G is the holomorph of H, and is a complete group if H is of odd order.† In the proofs of the various theorems of the paper free use is made of properties of rational integers, which are either quoted or established

21. In this paper in an elementary way without the assumption of Fourier's series, Professor Roe derives the two developments

$$f(x) = f(a) + \sum_{r=1}^{r=\infty} \int_a^x (-1)^{r-1} \frac{\sin(2rf'(x))}{r}\, dx,$$

where the range of a and x is determined by

$$\pi/2 \geqq f'(x) \geqq -\pi/2;$$

and

$$f(x) = f(a) + \frac{\pi}{2}(x-a) - \sum_{r=1}^{r=\infty} \int_a^x \frac{\sin(2rf'(x))}{r}\, dx,$$

where the range of a and x is determined by

$$\pi \geqq f'(x) \geqq 0.$$

In both cases the range is varied by suitable transformations. Numerous applications to the development of functions and the summations of certain series are given.

22. This paper is a continuation and further elaboration of matters suggested in Professor Roe's previous paper, presented at the October, 1916, meeting.

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* Miller, *Philosophical Magazine*, vol. 231 (1908), p. 224.
† Miller, *Mathematische Annalen*, vol. 66 (1908), p. 135.

The interpretation of the complex values of y in $y = f(x, z)$, surface, is given, and it is shown how the equations of surfaces may be expressed by the method considered, and how the equations of families of spirals in two coordinates on surfaces are derived. The spiral systems arising from the complex values of y in $y = \{x^{x-1}(x-1)^x\}^{1/(1-2x)}$ together with the value systems of this function are considered in detail. Special illustrations are given.

23. Professor Roe gives in this paper from an elementary standpoint the necessary and sufficient conditions for developing integral functions into products, and thus furnishes a method for developing into products series whose zeros are known or can be found, the results of which agree with those found by the methods of the theory of functions. Illustrations are furnished.

24. In this paper Mrs. Roe continues the investigation begun in her paper presented at the October, 1916, meeting. From the product and quotient tables she derives new formulas for the coefficients of each of the complexes involved, in terms of each of the others, together with new proofs of symmetry in certain of the tables of such coefficients.

25. For the Makeham approximation or graduation the English H^m Mortality Table was divided at the age 28. Professor Dodd finds it convenient to divide the American Experience Table at the age 70, and to approximate each portion separately by means of a more simple function, viz.,

$$a + br^x + cs^x.$$

This function gives very closely the number of survivors at the age x. The terms before and after multiplication by the powers of the discounting factor give rise to geometric progressions, making easy the computation of the life functions—including increasing life functions—at any desired rate of interest. This assists in the problem of determining the interest rate which the insured actually realizes, on the basis of the gross premium. In constructing or graduating a table by the above function from raw material, the least square method may be employed readily, after first approximations have been obtained.

26. The theorem of the note by Professor Gillespie is: the repeated integral of the bounded function $f(x, y)$ may be taken in either order with equal results; i. e.,

$$\int_0^1 dx \int_0^1 f(x, y)dy \quad \text{and} \quad \int_0^1 dy \int_0^1 f(x, y)dx$$

are equal where they both exist.

27. Professor Hurwitz presents an analogue to the Sturm-Liouville theory, leading to the expansion of two given functions in terms of solutions of a pair of linear ordinary differential equations with linear parameter, satisfying certain linear boundary conditions, the coefficients in the two expansions being the same.

28. In this note Professor Graustein proves the following theorem in connection with Volterra's theory of isogenous complex functions of curves: If the gradients c_1, c_2 of the functions F_1, F_2 in $F = F_1 + iF_2$, where $F_1 = F_1|[L]|$, $F_2 = F_2|[L]|$ are functions of the first degree of the space curve L, are in general analytic, the gradients γ_1, γ_2 of Φ_1, Φ_2 in $\Phi = \Phi_1 + i\Phi_2$, an arbitrary complex function of L of the first degree isogenous with F, are analytic save perhaps in points of singularity of c_1 or c_2 and points in which both these vectors are indeterminate.

29. Heine's proof of the existence of the functions of the elliptic cylinder, that is, of the existence of real periodic solutions of period 2π of the differential equation

$$\frac{d^2E(\varphi)}{d\varphi^2} + 4\left(\frac{2}{b}\cos 2\varphi + z\right)E(\varphi) = 0, \qquad b \neq 0,$$

—a differential equation fundamental in physical problems dealing with elliptic cylinders and elliptic membranes—is clearly at fault in an essential point. Dannacker's attempt to remedy the error is unsatisfactory. In the present paper Dr. Curtis sets up the proof in a rigorous fashion.

30. In Professor Eiesland's paper an attempt has been made to develop a geometry of flats in odd n-space on the same lines as Plücker's geometry in 3-space. Rectangular coordi-

nates are used throughout and some interesting results are obtained. The subject has of necessity led to certain projective properties of n-space which are believed new. The flat-complex seems to contain far-reaching possibilities when completely developed. One thinks especially of the interpretation of flat-geometry as sphere-geometry on the one hand and on the other as anallagmatic geometry in the next higher space, such as has been treated by Professor Ranum.

31. Professor Ettlinger generalizes the Bôcher-Sturm oscillation theorem to the following system:

$$d[K(x, \lambda)u_x(x, \lambda)]/dx - G(x, \lambda)u(x, \lambda) = 0,$$

$$L[a, u(a, \lambda)] = 0, \quad L[b, u(b\ \lambda)] = 0,$$

where

$$L[x, u(x, \lambda)] = \alpha(x, \lambda)u(x, \lambda) - \beta(x, \lambda)K(x, \lambda)u_x(x, \lambda).$$

In general the same conditions are placed on the coefficients of the differential equation and the boundary conditions* except that $\alpha(a, \lambda)/\beta(a, \lambda)$ and $-\alpha(b, \lambda)/\beta(b, \lambda)$ decrease as λ increases, and if $\beta(x, \lambda) \not\equiv 0$ and

$$\{\alpha\beta\} = \alpha_x(x, \lambda)\beta(x, \lambda) - \alpha(x, \lambda)\beta_x(x, \lambda) + \alpha^2(x, \lambda)/K - \beta^2(x, \lambda)G,$$

then $\lim (\lambda \to \Lambda_1) \max \{\alpha\beta\} < 0$ and $\{\alpha\beta\}$ changes sign once and only once as λ increases. The existence of an infinite set of characteristic numbers for this system is proved and oscillation theorems for $L[x, u_p(x)]$ and $u_p(x)$ are established. The Sturm-Bôcher theorem appears as a special case of this result, where a particular form is assigned to $\alpha(x, \lambda)$ and $\beta(x, \lambda)$.

32. In a forthcoming paper in the *Transactions* Professor Ettlinger extends the work of Birkhoff concerning the solutions of a general self-adjoint linear system of the second order. The object of the present paper is to generalize these results to a system with more general boundary conditions, and to derive more exact oscillation theorems.

Consider the second order linear differential equation

* See Bôcher, Leçons sur les Méthodes de Sturm, Paris, 1917, p. 63.

$$d[K(x, \lambda)u_x(x, \lambda)]/dx - G(x, \lambda)u(x, \lambda) = 0,$$

$$L_i[a, u(a, \lambda)] = L_i[b, u(b, \lambda)] \qquad (i = 1, 2),$$

where

$$L_i[x, u(x, \lambda)] = \alpha_i(x, \lambda)u(x, \lambda) - \beta_i(x, \lambda)K(x, \lambda)u_x(x, \lambda).$$

Under the corresponding sets of conditions, which are the natural extensions of those imposed in the earlier paper, the existence of an infinite set of parameter values for this system is established and an exact oscillation theorem obtained for $L[x, u_p(x)]$ and a slightly less exact theorem for the oscillations of $u_p(x)$.

The case considered above yields the other (see the earlier paper) by assigning a special form to the coefficients of the boundary conditions.

The work is based on the generalized Sturmian theorem of oscillation (see the preceding abstract).

33. As a result of an examination of the definition of mass in a large number of current books on mechanics and physics, Professor Huntington finds the following principal ways in which the mass of a body is said to be measured: (1) by force per acceleration (inertia); (2) by the beam balance (standard weight); (3) by mutual acceleration (Mach's definition); (4) by the number of supposed identical particles in the body. The pseudo-definition of mass as "quantity of matter," and various circular definitions, are condemned. The conditions for measurability in general are stated, and suggestions in regard to the best use of the term mass are discussed. The paper will be offered to the *American Mathematical Monthly*.

34. Professor Eisenhart's address, delivered before a joint session of the Society and the Mathematical Association, was devoted to an analysis of Darboux's contributions to geometry. This address will be published in the January issue of the BULLETIN.

F. N. COLE,
Secretary.

IRRATIONAL TRANSFORMATIONS OF THE GENERAL ELLIPTIC ELEMENT.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society, April 28, 1917.)

In 1865 G. G. A. Biermann published as derived from Weierstrass's lectures the following formula:

$$(1)\quad F(x) = x_0 + \frac{\sqrt{R(x_0)}\sqrt{S} + \frac{1}{2}R'(x_0)[s - \frac{1}{24}R''(x_0)] + \frac{1}{24}R(x_0)R'''(x_0)}{2\left[s - \frac{1}{24}R''(x_0)\right]^2 - \frac{1}{2}A\cdot R(x_0)}.$$

F is the solution of

$$(2)\quad (F')^2 = AF^4 + 4BF^3 + 6CF^2 + 4B'F + A' = R(F).$$

The accents used with F and R denote differentiation, x_0 is an arbitrary constant, and A, B, C, B', A' are constant coefficients. Also

$$(3)\quad S = 4s^3 - g_2 s - g_3 = 4(s - \epsilon_1)(s - \epsilon_2)(s - \epsilon_3),$$

$$s = \wp(x), \quad g_2 = AA' + 3C^2 - 4BB',$$

$$g_3 = ACA' + 2BCB' - AB'^2 - A'B^2 - C^3.$$

In Enneper, Elliptische Functionen, Greenhill, Elliptic Functions; and Haentzschel, Reduction der Potentialgleichung, are several applications of this formula.

Replacing x by $t + iu$ and F by $x + iy$ in the preceding equations gives

$$(4)\quad x + iy = \varphi(t + iu),$$

from which may be obtained in the usual manner two mutually orthogonal families of curves having t and u as parameters. For several cases this has been done by Haentzschel, and the writer has given discussions and extensions in *Archiv der Mathematik und Physik*. It is evident that (1) becomes very simple when x_0 is a root of $R(x) = 0$, for which Haentzschel's

results may be consulted. When x_0 is infinite Enneper obtains from (1)

$$(5) \qquad v = \frac{\sqrt{A}\ \sqrt{S} + 2Bs - BC + AB'}{2(B^2 - AC) - 2As},$$

but does not develop the results. However, the rationalization of (5) gives

$$(6) \quad s^2 - s(Av^2 + 2Bv + C) + (B^2 - AC)v^2 + (BC - AB')v + (C^2 - AA')/4 = 0.$$

Next placing x equal to 0 in (1),

$$(7) \quad s^2v^2 - s(A' + 2B'v + Cv^2) + (B'^2 - A'C) + (B'C - A'B)v + (C^2 - AA')v^2/4 = 0,$$

which becomes (6) when v is replaced by $1/v$, noticing the interchange of constants required by (2). Later in this paper reference will be made to (7) in another discussion.

In using (1) to obtain orthogonal curves it is of great advantage to make the following changes: $F(x)$ is replaced by $x_0 + v$ and (2) is written

$$(8) \quad (v')^2 = av^4 + 4bv^3 + 6cv^2 + 4dv + e = R(x_0 + v) \qquad [v = x' + iy'],$$

whence

$$(9) \quad a = \frac{R''''(x_0)}{4!} = A, \quad 4b = \frac{R'''(x_0)}{3!}, \quad 6c = \frac{R''(x_0)}{2!},$$

$$4d = R'(x_0), \quad e = R(x_0),$$

$$g_2 = ae - 4bd + 3c^2, \quad g_3 = ace + 2bcd - ad^2 - eb^2 - c^3.$$

The rationalized form of (1) thus becomes

$$(10) \quad s^2v^2 - s(cv^2 + 2dv + e) + (d^2 - ce) + (cd - be)v + (c^2 - ae)v^2/4 = 0,$$

which is of the same form as (7).

Greenhill, on page 151 (l. c.), refers to (1) as a method of reducing the general elliptic element to the Weierstrass form. Now in using (1) as a solution of (2) it is required that s shall satisfy

$$(11) \qquad (s')^2 = 4s^3 - g_2s - g_3,$$

so that one value is of course $\wp(x)$. Since (11) is only a special case of (2) the corresponding form of (1) gives the solution

$$\bar{s} = \frac{\sqrt{(-g_3)}\,\sqrt{S} - \frac{1}{2}g_2 s - g_3}{2s^2} \tag{12}$$

or in rational form,

$$s^2\bar{s}^2 + g_2 s\bar{s}/2 + g_3(s+\bar{s}) + g_2^2/16 = 0. \tag{13}$$

By elimination of s between (10) and (13) there is obtained a new equation involving v and $\bar{s}$ which is a still more complicated solution of (8), or in other words a method of reducing the general elliptic element to the Weierstrass form. Because v and $\bar{s}$ do not enter symmetrically this equation is too complicated to be of interest.

The conjugate of (10) formed by changing s and v into σ and w respectively provides a transformation of the general into the Weierstrass form and vice versa, so that making s and σ identical, and then eliminating s gives an identical transformation of the general elliptic element. After cancellation of the trivial factor $(v-w)^2$ the result is

$$\begin{aligned} &v^2w^2(b^2e^2 - aed^2) + vw(v+w)(3bce^2 - 2bed^2 - ade^2) \\ &\quad + vw\left(\frac{9c^2e^2}{2} - 2bde^2 - 2d^4 - \frac{ae^3}{2}\right) \\ &\quad + (v+w)(3cde^2 - be^3 - 2d^3e) \\ &\quad + (v^2+w^2)\left(\frac{9c^2e^2}{4} - 3cd^2e + d^4 - \frac{ae^3}{4}\right) = 0. \end{aligned} \tag{14}$$

If s and σ in (10) and its conjugate are written $\wp(t+iu)$ and $\wp(t-iu)$ respectively, and u is eliminated, the resulting equation of the sixteenth degree is resolvable into one factor of the eighth degree and two equal factors each of the fourth degree having $\wp(2t)$ as a parameter. Now if $\wp(2t)$ is taken as $(d^2-ce)/e$ the equal factors are identical with (14), which also, as is evident from the preceding, corresponds with $s-\sigma=0$. Lastly the discriminant of (14) as a quadratic equation in v is

$$\begin{aligned} &e(aw^4 + 4bw^3 + 6cw^2 + 4dw + e)(d^2 - ce - e\epsilon_1) \\ &\qquad \times (d^2 - ce - e\epsilon_2)(d^2 - ce - e\epsilon_3). \end{aligned} \tag{15}$$

UNIVERSITY OF PENNSYLVANIA,
April, 1917.

NOTE ON THE PARAMETRIC REPRESENTATION OF AN ARBITRARY CONTINUOUS CURVE.

BY PROFESSOR DUNHAM JACKSON.

(Read before the American Mathematical Society, April 28, 1917.)

A CONTINUOUS curve may be defined by a pair of equations

$$x = f(t), \qquad y = \varphi(t),$$

where the functions f and φ are defined and continuous throughout some interval, say for $0 \leqq t \leqq 1$, and are not both constant throughout that interval. It is sometimes an aid to the imagination to suppose that they are not simultaneously constant throughout any subinterval of their interval of definition. The question naturally arises, whether this implies a restriction on the curve itself, or merely on the particular parametric representation employed. It is obvious that if there are only a finite number of intervals where f and φ are constant together, a suitable change of parameter will eliminate such intervals entirely. The present note offers a proof that this is true in all cases. A decidedly different proof has been given by Fréchet;* it is believed that the method of treatment here adopted is considerably more elementary than his, as far as obtaining the indicated result is concerned, though it does not yield certain additional knowledge which is incidental to his presentation.

On careful scrutiny, it appears that the problem of a satisfactory definition of the identity of two curves is not an altogether simple one;† to avoid a discussion of this point, the result is stated in the following form, which is free of ambiguity:

THEOREM. *Let a pair of functions*

$$x = f(t), \qquad y = \varphi(t) \tag{1}$$

be defined and continuous for $0 \leqq t \leqq 1$. *Then it is possible to*

* Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, vol. 22 (1906), pp. 1–74; pp. 58–59, 67–70.

† Cf. Fréchet, loc. cit., pp. 51–53.

define a function $\sigma(t)$, single-valued, continuous, and non-decreasing for $0 \leqq t \leqq 1$, and a pair of functions $F(\tau)$, $\Phi(\tau)$, single-valued and continuous for $\sigma(0) \leqq \tau \leqq \sigma(1)$, so that the two latter functions are not simultaneously constant throughout any interval $\tau_1 \leqq \tau \leqq \tau_2$, where $\tau_1 < \tau_2$, and so that

$$F(\sigma) = f(t), \qquad \Phi(\sigma) = \varphi(t),$$

identically for $0 \leqq t \leqq 1$.

The function σ will be so chosen that $\sigma(0) = 0$, whereby the interval of values for σ becomes $(0, \lambda)$, if $\lambda = \sigma(1)$; but this remark is unimportant, to the extent that the possibility of making such a choice of σ is clearly implied in the theorem as already stated, since the subtraction of a constant from $\sigma(t)$ does not affect any property of it with which the theorem is concerned; and it would be possible further to replace the value λ by 1.

As a consequence of the theorem, it is to be observed that within the limits $0 \leqq t \leqq 1$, $0 \leqq \sigma \leqq \lambda$, to every value of t corresponds one (and just one) value of σ, and to every value of σ corresponds at least one value of t, so that every point (x, y) given by the relations (1) is given by the equations

$$x = F(\sigma), \qquad y = \Phi(\sigma),$$

and vice versa. But it is not true in general that t is a single-valued function of σ.

To prove the theorem, let us notice first that it is practically obvious if the curve (1) is rectifiable, the arc-length s, defined in the usual way as limit of inscribed polygons, serving the purpose of σ. The function $s(t)$ is certainly single-valued, continuous, and non-decreasing.* To every value of s between 0 and l, where l is the total length of the curve, corresponds at least one value of t, in consequence of the properties just named, and so at least one point (x, y) of the curve. Furthermore, although t need not be a single-valued function of s, a specified value of s can not give more than one point (x, y), since the values of s for any two distinct points differ by at least the length of the chord joining the points. So x and y are single-valued functions of s, which may be denoted by $F(s)$ and $\Phi(s)$, defined for $0 \leqq s \leqq l$, and fulfilling the identities $F(s) = f(t)$, $\Phi(s) = \varphi(t)$, from the manner of their

* Cf., e. g., Pierpont, Theory of Functions of Real Variables, vol. 2, p. 585.

definition. They are continuous, since $\Delta x \leqq \Delta s$, $\Delta y \leqq \Delta s$; and they are evidently not constant together over any interval. A proof of the last statement, labored as the proof of an evident fact is likely to be, is as follows: Because of the monotone character of s, an interval of values of s corresponds to an interval of values of t. If F and Φ were constant throughout an s-interval, f and φ would be constant throughout the corresponding t-interval; but then this t-interval would give just a single point (x, y), and so a single value, not an interval of values, of s. So the theorem is established for any rectifiable curve.

Now consider an arbitrary continuous curve, given in the form (1). Let $\omega(\delta)$ be the maximum of

$$\sqrt{[f(t'') - f(t')]^2 + [\varphi(t'') - \varphi(t')]^2}$$

for $0 \leqq t' \leqq 1$, $0 \leqq t'' \leqq 1$, $|t' - t''| \leqq \delta$. This function is continuous and non-decreasing for all values of $\delta \geqq 0$. The latter property is evident; the former follows from the facts that

$$\lim_{\delta=0} \omega(\delta) = 0, \qquad \omega(\delta_1 + \delta_2) \leqq \omega(\delta_1) + \omega(\delta_2).$$

The function $\omega(\delta) + \delta$ is continuous and increasing. Let ψ be the inverse function, so that

$$\delta = \psi[\omega(\delta) + \delta];$$

then ψ also is single-valued, continuous, and increasing, and $\psi(0) = 0$.

Let t be any point in the interval $(0, 1)$. Let $t_1, t_2, \cdots, t_{n-1}$ be any $n - 1$ points between 0 and t, arranged in order, and let $t_0 = 0$, $t_n = t$. Let

$$\delta_i = \sqrt{[f(t_i) - f(t_{i-1})]^2 + [\varphi(t_i) - \varphi(t_{i-1})]^2} \quad (i = 1, 2, \cdots, n),$$

and, finally, let

$$\sigma(t) = \text{upper limit of } \sum_{i=1}^{n} \psi(\delta_i) \tag{2}$$

for all possible choices of $t_1, \cdots, t_{n-1}$ and all values of n. This function $\sigma(t)$ will be shown to fulfil the requirements of the theorem.

In the first place, $\sigma(t)$ is defined for $0 \leqq t \leqq 1$. For

$$\delta_i \leqq \omega(t_i - t_{i-1}) < \omega(t_i - t_{i-1}) + (t_i - t_{i-1}),$$

and consequently

$$\psi(\delta_i) < t_i - t_{i-1}$$

and

$$\sum_{i=1}^{n} \psi(\delta_i) < t,$$

so that the upper limit in (2) is finite. It is evident that $\sigma(t)$ can not diminish when t increases.

Let $\sigma(t', t'')$ denote the upper limit of the sums $\Sigma\psi(\delta_i)$ obtained by subdividing the interval (t', t''); in particular, $\sigma(0, t) = \sigma(t)$. The functions σ with one and two arguments respectively will not be in any danger of confusion. The proof that $\sigma(t)$ is continuous resembles the corresponding proof for rectifiable curves,* but it is necessary to exercise some care, for two reasons. In the first place, a sum $\Sigma\psi(\delta_i)$ may be *diminished* by introducing new points of subdivision and retaining the old ones; it may even happen that the greatest possible value of the sum is attained by taking $n = 1$, and in no other way.† Secondly, if $t' < t'' < t'''$, while it is evident that

$$\sigma(t', t''') \geqq \sigma(t', t'') + \sigma(t'', t'''),$$

the sign of equality need not hold.‡

Let c be any point in the interval $0 < c \leqq 1$. It will be shown first that $\sigma(t)$ is continuous on the left at the point c. Since σ is monotone increasing, it is required only to show that if ϵ is any positive quantity, there is a $t < c$ such that

$$\sigma(t) > \sigma(c) - \epsilon. \tag{3}$$

Take any subdivision of $(0, c)$ such that the corresponding sum $\Sigma\psi(\delta_i)$ is greater than $\sigma(c) - \frac{1}{2}\epsilon$. For $t > t_{n-1}$, let

$$\delta_n' = \sqrt{[f(t) - f(t_{n-1})]^2 + [\varphi(t) - \varphi(t_{n-1})]^2}.$$

If t is sufficiently near to c, δ_n' will differ so little from δ_n that§ $\psi(\delta_n') > \psi(\delta_n) - \frac{1}{2}\epsilon$, and the points $t_1, \cdots, t_{n-1}$ give a subdivision of $(0, t)$ such that the corresponding sum $\Sigma\psi(\delta_i)$ exceeds $\sigma(c) - \epsilon$. Hence (3) is satisfied.

In the same way, it may be shown that, for $0 \leqq c < 1$,

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* Cf., e. g., Lebesgue, Leçons sur l'Intégration, pp. 53–54, 60.

† E. g., let $f(t) = \sqrt{t} - t$, $\varphi(t) = 0$. Then $\omega(\delta) = \sqrt{\delta} - \delta$, $\psi(\delta) = \delta^2$, and $\sigma(1) = 1 = \psi(1)$, while $\Sigma\psi(\delta_i) < 1$ if $n > 1$.

‡ In the example of the preceding footnote, $\sigma(0, \frac{1}{2}) + \sigma(\frac{1}{2}, 1) = \frac{1}{2}$.

§ Of course it is not inconceivable that $\psi(\delta_n') > \psi(\delta_n)$.

$$\sigma(t, 1) > \sigma(c, 1) - \epsilon,$$

if t is greater than c and sufficiently near to c. Since

$$\sigma(c, 1) \geqq \sigma(c, t) + \sigma(t, 1),$$

it follows that

$$\sigma(c, t) < \epsilon;$$

but even now it can not be inferred immediately (except for $c = 0$) that $\sigma(t)$ is continuous on the right at c, because of the failure of the additive property for consecutive intervals.

Let ϵ be any positive quantity. Let $t > c$ be so close to c that $\sigma(c, t) < \frac{1}{3}\epsilon$, which is possible, in consequence of the preceding paragraph; and so close also that the length of the chord joining the points $[f(c), \varphi(c)]$ and $[f(t'), \varphi(t')]$ is less than $\frac{1}{3}\epsilon$ for $c \leqq t' \leqq t$. If δ' and δ'' are the lengths of the chords joining these two points with any third point, then $|\delta'' - \delta'| < \frac{1}{3}\epsilon$; and it follows that

$$|\psi(\delta'') - \psi(\delta')| < \tfrac{1}{3}\epsilon, \tag{4}$$

since any increment of $\omega(\delta) + \delta$ is greater than or equal to the corresponding increment of δ, because of the monotone character of ω, and any increment of ψ consequently is less than or equal to the corresponding increment of its argument.

After choice of a particular value of t satisfying the conditions just specified, take any subdivision of the interval $(0, t)$ such that the corresponding sum $\Sigma\psi(\delta_i)$ exceeds $\sigma(t) - \frac{1}{3}\epsilon$, the points of division being $t_0 = 0, t_1, \cdots, t_{n-1}, t_n = t$. Suppose k is the index such that $t_{k-1} < c \leqq t_k$. Then the terms $\psi(\delta_{k+1}) + \cdots + \psi(\delta_n)$ constitute a sum which can not exceed $\sigma(c, t)$, and so is less than $\frac{1}{3}\epsilon$. If δ_k' is the length of the chord joining $[f(t_{k-1}), \varphi(t_{k-1})]$ and $[f(c), \varphi(c)]$, then

$$\psi(\delta_k) - \psi(\delta_k') < \tfrac{1}{3}\epsilon,$$

in consequence of (4). Hence

$$\psi(\delta_1) + \cdots + \psi(\delta_{k-1}) + \psi(\delta_k') > \psi(\delta_1) + \cdots + \psi(\delta_n) - \tfrac{2}{3}\epsilon > \sigma(t) - \epsilon,$$

and it follows that

$$\sigma(c) > \sigma(t) - \epsilon,$$

which proves the continuity of $\sigma(t)$ on the right at c.

The rest of the proof of the theorem is almost immediate. In consequence of the continuity of $\sigma(t)$ there corresponds to every value of σ between 0 and $\lambda = \sigma(1)$ at least one value of t, and so at least one point of the curve. There can not be more than one point for a single value of σ, because the values of σ corresponding to any two distinct points differ by at least $\psi(\delta)$, if δ is the length of the chord joining the points. If we set $x = F(\sigma)$, $y = \Phi(\sigma)$, these are single-valued functions of σ, and are identically equal to $f(t)$ and $\varphi(t)$ respectively, by their very definition. They are continuous, because

$$\Delta\sigma \geqq \psi(\sqrt{\Delta x^2 + \Delta y^2}), \qquad \Delta x \leqq \sqrt{\Delta x^2 + \Delta y^2} \leqq \omega(\Delta\sigma) + \Delta\sigma,$$

and similarly for Δy. Finally they are not constant together in any interval, by the same argument as in the case of a rectifiable curve.

The extension of the work to curves in three or more dimensions requires only the writing down of a correspondingly larger number of symbols.

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JOHN WALLIS AS A CRYPTOGRAPHER.

BY PROFESSOR DAVID EUGENE SMITH.

IT is not probable that many bibliophiles in the domain of mathematics, seeing upon their shelves the sumptuous tall copies of the Opera Mathematica* of John Wallis, and consulting their noteworthy historical chapters, the first serious effort in the history of mathematics in England, ever consider that the author was one of the world's greatest decipherers of cryptic writing. To be sure his biographies give us the information that he was interested in cryptography, but the extent of this interest, the sixty years devoted to the subject, the services rendered to the State, the rewards and disappointments that came to him as a result—of all this the biographies tell us practically nothing. It is partly because of this fact,

* Oxoniæ, MDCXCV.

partly because of the interest naturally excited by the present war, but chiefly because of the new light that certain letters, hitherto unpublished, throw upon the life and character of a mathematician of merit, that this article is written.

After Wallis (1616–1703) had received the bachelor's degree at Emmanuel College in 1637 and the master's degree three years later, he became chaplain (1642) to the widowed Lady Vere. While so engaged there was brought to him a letter in cypher relating to the capture of Chichester.* In two hours he succeeded in deciphering it, and thus began his career as an expert in cryptography. As a reward for his services in this line he was given (1643) the living of St. Gabriel in London, and later (1647) that of St. Martin. His Savilian professorship (1649), his office of Keeper of the University Archives, and his chaplaincy to the king (1660), did not interfere with his work in the field of secret writing, and such was his standing that he was employed by the Earl of Nottingham as decipherer for William III. Loyal to his country, he steadily refused the importunities of Leibnitz to reveal his methods, but he made them known to his own son,† who lacked, however, the father's genius in this as in other respects, and to his grandson, William Blencow.‡

Some years ago, in working among the early mathematical manuscripts in the British Museum, I came across a considerable number of letters written by Wallis, many of them relating to this work in cryptography. While it is not feasible to publish them all, it may not be without interest to give a few of those which reveal the character of Wallis the man as well as of Wallis the cryptographer.§

The loyalty of Wallis to his own country, and his desire to have the work which he had begun for the state continue after his death, are seen in the following letter:

For Mr Tilson, at

Sir, Oxford, Mar. 20, 1700/1.

I am obliged to you (though unknown) for your favour in drawing up the Warrant to be signed by his

* Dec. 27, 1642.

† John Wallis, b. Dec. 26, 1650, B.A., of Trinity College, Oxford (1669), called to the bar in 1676.

‡ He was the son of Sir John Blencow (b. 1642) who married the daughter of Wallis in 1675.

§ These letters are in Add. MS. 32499, entitled "Letter Book of Dr. John Wallis, 1651–1701."

Majesty for a Pension of 100$^{lb.}$ pann^{m}. payable to me, in order to my instructing a young man (William Blencow, son of Mr Justice Blencow) in the Art of Deciphering.

Concerning which I have this further to suggest; viz. That though it was his Mat^{ies} chief design in that grant, partly to gratify me for services allready done, and partly to incourage my propagating of that Art, that it do not dy with me; I think yet he may be inclinable allso to incourage the young man to diligence therein; and perhaps may be gratiously pleased to order the pension payable to me during my life, (which is not like to be long (being now in my 85th year), and thenceforth to the young man during his Majesties pleasure.

I would have suggested this to my Lord Godolphin, or S^{r} Stephen Fox, but that I have not the honour to be personally acquainted with either of them. But if this may not be obtained, then to have it pass as it is now drawn.

I may add this allso; That I have been solicited by Mijn Heer Leibnitz, more than once, in behalf of the Elector of Hanover: who is willing to send hither some young men, whom he desires I would instruct therein; leaving it to me to make my own proposals on what terms I would undertake it. To which I have returned answere, That I shal be ready, my selfe to serve his Electoral Highness if there be occasion: but the skill of doing it, being a curiosity which may be of use to my own Prince, I do not think it proper to send it abroad, without his Ma^{ties} leave.

If you shal think it proper (in such way as you may judge most fit) that this be layd before his Ma^{tie}, you may therein oblige,

S^{r} Your humble servant

JOHN WALLIS.

Of his painstaking work in deciphering, the following letter addressed to R. Hampden* gives evidence:

* Richard Hampden (1631–1695) was a son of John Hampden (1594–1643) of revolutionary fame. At this time he was representing Wendover and was prominent as a spokesman of the Whigs.

For R. HAMPDEN ESQ^{r}, at the house of Comons.

S^{r} OXFORD, Aug. 3. 1689.

In a pacquet from the Earl of Nottingham, I lately received inclosed a Letter from you to which (in a like pacquet to his LO^{p}. by the last post) I returned a short answere. But should have been more particular were it not for fear of loosing that Post; which was within a few minutes of going before I had dispatched that pacquet as it was; (though I had taken true pains about it, both by day, & by night, because you desired to have it done speedyly).

I find, in the Cipher, there had been some mistakes committed: As (in one place) *Brigadies* for *Brigadier* (where 36, had been miswritten for 35;) and *efget* for *effet* (twice;) and *pued* for *pied;* and *alller* for *aller:* All which (without altering the original, which I was not to do) I tooke the liberty to correct in my transcript. Beside which, I do not know that I have varied a letter from what was in Cypher, save that in one place, I doubt I have miswritten *pouroit* for *pouvoit.*

Some few words I could onely supply from the sense; As (in the second or third line) *c . . . . s,* of which I was sure (that is, that it was a word which begins with *c,* to which *s* was added in the end) for which (from y^{e} sense), I supply *chances* (because I could not think of any other such word which would make good sense, the same figure occurs not again.) In like manner *sans, tout, homme,* (& some few other) but these are confirmed by their so occurring several times in different circumstances; of which therefore I make no doubt.

For some few (of which I had some doubt) I have left blanks; as (in the second line) *entraine t . . . . . .* (where I think *train* or *traict* or some such word is to be supplied.*

Soon after, there is *c . . . . toute sorte de r . . . .* which is I presume to be read *contre toute sorte de regle:* which occurs again in y^{e}. next line, where *regle* is spelled by letters; unless in the former place we choose to read *raison.*

Some other French words I mett with, which I did not understand but I give them as I find them: & in some

* No; but tousiours [so in margin of original].

places there seems to me some (sic) a word wanting (which yet I would not take upon me to supply) as, where *donner* doth (two or three times) occur for *ordonner;* (to *order* or *give order*), but perhaps it is onely my being unacquainted with y.e French way of speaking.

Mr d'Hamilton and *Mr Rose*, I do not know whether I am to read *Monsieur* or *Monseigneur* (because I do not know the quality of the persons;) But I think ye. latter; because I find the same title afterwards given to the person to whom he writes, wch (as appears by ye beginning & end of the letter) must there be read *Monseigneur*. But *M. . . . Melfort* (where another figure is used) is, I presume, to be read *My Lord Melfort*. But I chose to write *Mr d'Hamilton* &c because in the beginning of the letter (which is not in cypher) I find written *Mr d'Hamilton*.

Non estoit parfaitement b. . . . I suppose is to be supplyed by *bas* or some other such word beginning with *b*.

En e. . . . de tout craindre, is to be supplied by *exces*, or some such word beginning with *e*.

Ny a. . . . pas ici, is to be supplyed *avons*.

L'escrit m. . . . apres is to be supplied by *mesme*, or some other word beginning with *m*.

And, near the end, *e. . . . e*, is certainly to be read *envoyé*, but the word not then coming to mind, I leaft a blank.

Of all the rest, I think myself very secure not to have missed any thing.

When he says, there were killed ore wounded, *about fourty officers & some sixty soldiers;* it seems strange that, if 40 officers, there should not be more than 60 soldiers; unless *soixante* (60) be miswritten for *six cents* (600;) but I give you as it is.

The name subscribed, I know not wheither it (sic) *Mointis* or *Nointis* or *Pointis;* For it is so written as it may bear any of them. I take it to be *Nointis*. But it is surely the commander in chief there.

Nor do I know to whom it is written (because the cover was not sent me,) but it must be somebody much superior to him that wrote it; as appears by the manner of writing.

Nor know I how it becomes to be so rotten and discoloured in so short a time: Unless possibly it may have been thrown overboard into salt water & recovered from thence.

I know not whether it may not be thought fit to have it translated into English & printed.

S^{r}. you will please to excuse my giving you. the trouble of so long a letter about such little things. But I was willing to let you see how far I could conclude certainly, & where but by conjecture; & upon what kind of grounds I did so conjecture. Those perhaps who are better acquainted with the language, may, in these cases, conjecture better, than

S^{r}

Your very humble servant

JOHN WALLIS.

Some idea of the importance of the communications that came into his hands may be obtained from a letter addressed to the Earl of Nottingham, then one of the secretaries of state with charge of the war department.* The letter is as follows:

For the EARL OF NOTTINGHAM at Whitehal.

My Lord, OXFORD, Aug. 18, 1689.

I am allmost ashamed to tell yo^{r}. Lordship how much time & pains & study I have employed on that very perplexed cypher in the Letter from Poland, & have not yet dispatched it. But by what I have done all ready, I find two things (which seem to me) of moment. One is a Treaty (or intreaty rather) of the French King with the King of Poland presently to make a war on Prussia.† The other, about a marriage of the Princess of Hanover with the Prince of Poland, promoted by the French King. How far it may be of concernment to us to know it, I am no competent Judge; But I had thought it did become me to give this timely notice of it (lest there might be a prejudice by delay) while I am preparing to give a fuller account of that letter (than yet I can), by the next Post (or that next after it) if I have not in the mean time order to come up with it myself. who am

My Lord Your honours very humble servant

JOHN WALLIS

* This was the second Earl of Nottingham (1647–1730), a great Tory but an anti-Jacobite.

† To give a diversion to the Emperour. [So in the original, as a note at the end of the letter.]

Many of the letters show Wallis as a man willing to serve the King but always seeking for rewards for himself and all his family. The following gives some idea of his methods of ingratiating himself with the powers at Court, a procedure then undoubtedly less objectionable in the eyes of the world than now:

To y^e^ EARL OF NOTTINGHAM at Whitehall.

My Lord, OXFORD, Nov. 12, 1689.

I have ordered my son (who is now, or will be sudainly in London,) to wait on your Honour & acquaint you that I sent you a packet by last Sunday's Post; & two before by Sunday & Tuesday Posts next foregoing; which I hope came safe to yo^r^ Lo.^ps^ hand. But I am now at a stand, by reason that the five other letters are at lest in three (I think in four) different ciphers, & all new, which I have not had before. Besides which, I have myself (since my last) been very ill, so as not to be in a capacity to apply myself (as I would) to that or any other business which requires intentness; & continue stil indisposed. But I shal (as my strength & y^e^ difficulty of y^e^ work will permit) endeavour to serve his Majesty, & your Lordship therein.

I humbly thank yo^r^ Lordship for the kind expressions (of your readyness to oblige me) in yo.^r^ Lo.^ps^ letter of Octob. 24. I have not been fond of being a great man (studying more to be serviceable than to be great) & therefore have not sought after it. Especially in the late times, wherein I could safely foresee that divers things might then be expected from me, with which I could not comply. But I am capable of any promotion Ecclesiastical, or in a way of literature; & of my accomplishments for either, I had rather others should speak than myself. And I have a son, a Barrister of the Inner Temple who though his attendance on an estate he hath in the country have caused him to neglect the practise of y^e^ law. at London, yet he understands business very well, & is sagacious inough in it, & capable of any civil imployment; & of whom I could say more if he were not my son. And I hav a son-in-law (who married one of my daughters) M^r^. Serjeant Blencow (of y^e^. Inner Temple allso) an able Lawyer both at Chamber practise & the barr; & not inferior therein to divers of

those on the Bench; And neither of them in the lest wise sullyed by any unhandsome actings in the late bad times; but are very cordial to his Ma^ties^ interest. Any thing of kindness to either of them, will be so, to

My Lord, Yo^r^. Honours very humble servant

JOHN WALLIS.

Another letter, also showing the desire to be in the favor of the Court, and having some interest in view of the present European situation, is as follows:

To y^e^ EARL OF NOTTINGHAM at Whitehal.

My Lord, OXFORD Febr. 20, 1689/90. Thursday.

Before I left London (which was on Munday last) I was (on Saturday) to wait on yo^r^. Honour both at Berkshire-house, & at y^e^. Office in Whitehal, to receive your commands for Oxford, but was not so happy as to find you at either. And I had y^e^ same day waited on the Envoy of Brandenburg on the like occasion. Some while before which, a friend of mine dining at the Table of a Noble Lord, where divers Lords were present, & other persons of quality, one of them did openly declare, that y^e^ Elector of Brandenburg had sent me, for a present, a large medal, with an Inscription & a chain of gold (for deciphering French Letters &c). My friend sayd, He thought it was a mistake; for that he had spoken with me the night before, & I knew of no such thing. The other person of quality confirmed what he had before sayd, & went on to describe it so particularly as if he had seen it. Now (beside that this was not so) I doubt whether, if it were, it were advisable to make the thing so publike. That it was not so, I find, because I have since waited on that Envoye, & he sayd nothing to me to that purpose. He sayd indeed that his Master intended to send me a mark of his favour; but that none such was yet come; & added withal, that it is as the King's concern to gratify me for those services, as being a joint interest (sic) of him & his Allyes. And, why I doubt whether it be advisable to make it so publike (that I decipher y^e^ French Kings letters) is, because it can then hardly be avoided but that this will some way or other come to y^e^. French Kings knowledge from some of his correspondents here; which

will be attended at lest with one or both of these inconveniences, viz: A greater care to prevent the intercepting of such letters; And, a change of the ciphers they now use (which they have allready changed more than once, for I have allready nine or ten of their ciphers by mee) for others more difficult, & (perhaps) in superable. The two letters which I have brought with me from London, (one of two sheets, the other of five sheets) I shall endeavour to dispatch as soon as I get a little leisure. And am

My Lord, Yo^{r}. honours very humble servant

JOHN WALLIS.

That Wallis often felt that the rewards were not commensurate with the labor, and that he did not hesitate to make this fact known, may be seen from a letter written in 1691:

For M^{r} HARBORD, in St. James's park London.

Sir, Aug. 15, 1691.

I should begin my letter with thanks for yo^{r} civilities of late, & before; And that you have pleased to concern yourself for me without my knowledge, (and without any complaint of mine.) But (because your time is precious) I shall rather obey yo^{r} commands, in giving you some account of what you ask me.

It is now more than two years compleat, since by a letter from my Lord Nottingham & another from M^{r} Hampden, I was desired to decipher an intercepted French letter, from y^{e} camp then before London-Derry; and (soon after) a second from y^{e} same place; &, then, a third from Poland to the French King from his Ambassador there. Which I performed to that satisfaction, y^{t} his Lordship made me a Present (from y^{e} King, I suppose) of Fifty pound. Which I looked upon as a handsome gratuity for y^{e} service then done, & as a testimon of his Ma^{ties} acceptance (which I valued) & returned my acknowledgements accordingly.

After this I was kept in a continual imployment of like services: wherein I took true pains, (& my son with me, who was assistant to me;) by day & by night, (because letters came faster than we could well dispatch them:) And it was almost our whole imployment for a long time.

About a year after; having received nothing (but good

words) in requital (since y^t for y^e first three letters) I let M^r Hampden know so much (upon who's joint desire I had undertaken y^e service.) Upon who's speaking(I suppose) to my Lo. Nottingham, he sent me another 50^l, which is all I have received.

Perhaps you will judge this to be no great gratuity for so much service (which every body could not have done him:) Having by that time deciphered some hundred sheets of paper, sent me in cipher, for that purpose. And some of them, (he can tell you,) were of very great concernment: The deciphering some of those letters having quite broke all y^e French King's measures in Poland for that time; & caused his Ambassadors to be thence thrust out with disgrace. Which one thing was of much greater advantage to his Ma^{tie} & his Allies, than all that I am like to receive on that account.

I remember that my Lord Arlington (a little before he gave over being Secretary) sent for me on purpose, & did (without my asking) give me 50 Guinny's in hand; & promised me 50 guinney's a quarter (which were duely payd me) to endeavour the deciphering of such letters as should be sent me from time to time. And I had of him (as I remember) 200 Guinney's before he gave over y^e place of secretary. Yet I had not done for my Lord Arlington the tenth part of that service I have done my Lord Nottingham.

However; without disputing the point, whether that Fifty pound were a Noble recompense for all y^e services then done (except y^e three first letters) & for all that I was to do afterward (for that, it seems, is now the case:) I have continued the same trade for more than another full year, without having received (or being like to receive, for ought I see) y^e value of one penny.

'Tis true; I have had all along a great many good words; That he is my humble servant; My faithful servant; my very faithful servant; That he will not fail to acquaint y^e King with my diligence & success in this difficult matter; That he is very sensible of y^e pains I had taken in this troublesome work; That he shal be very glad of any opportunity of doing me any service: That y^e King is sensible of my zeal & good affection, and will (he beleeves) in a short time give me some mark of his favour (meaning,

I suppose, some preferment,) wherein his Lo^ps. endeavour shal not be wanting; That he will not be wanting to represent my services to my advantage. And y^e like.

But these Promises are of Two-years standing. And it may seem strange; if in all that time (while so many Deans, & Bishops, & Arch Bishops have been made) his Lo^p. could never find an opportunity of what he so much desired.

Now, y^e Apothecaries tell us, that some Cordials are y^e less Operative, for having been long made. And I have been told (by one who pretends to understand them) that Courtiers promises, if they do not operate quickly, it is not likely y^t they will operate at all, (but are to pass for words of course:) and that to depend upon stale promises, is like calling for an Old Debt; w^ch y^e longer it hath been forborn, grows y^e more desperate.

As to what he intimates of preferment: I have asked none, & they offer none, (& so far we are are agreed:) And there are so many who (by themselves or friends) seek preferment, y^t it is not like to fall upon one who looks not after it. And I easyly foresee so many exceptions, & Diversions, that may be made by persons otherwise concerned, (as, y^t one thing is too good for me; another, not good inough; a third, fit for somebody else, y^t is to be first served: and y^e like:) that I have no great expectation of it (being not acquainted with y^e methods of Court applications) nor much concerned for it: Having allways endeavoured rather to deserve preferment, than to have it: I may perhaps, in time, be offered something that may not be advisable for me to embrace, (because I must, for it, leave what I have in Oxford; which if not better than it; may at least be so to me;) & then it will be easy to object that he who doth not accept a little one, doth not deserve a better.

But having (for my Lord Nottingham) condescended to do clarks-work; I might at least expect clarks-wages; (without being thought mercenary or ungentile:) And I presume there is never a clark his Lo^p. keeps, but is (one way or other) better payd, for y^e work he doth, than I am.

He may say perhaps, This is (not his, but) the King's service. Very true. And so is all y^e service his Lo^p. doth as Secretary. Yet he is well payd for it. And, so

wel, that he may (out of his allowance) afford to gratify those y^t^ work under him.

All which is sayd, not that I do distrust their Ma^ties^ gratious inclinations; or, would or (sic) under value their favours. But, because Princes see with other mens eyes; & know no more of Persons or services than as they be represented. And then, if one man (by a friend at Court, who hath the Kings ear) be represented as a person of prodigious parts, of great learning, of a clear reputation, of great esteem at home & abroad (in all parts of y^e^ learned world,) as a credit to y^e^ University, an honour to the Nation; as a friend to the Government, & zealous for it; of a good judgement, a clear apprehension; that speaks well & writes well, in what he undertakes, & may (if imployed) be many ways serviceable to the Government; a man of temper & moderation: not bigotted to a faction (laying great weight upon little things, which cannot bear it, or deserve it;) who hath done, & can do services which others do not pretend to: (though there may be much of Hyperbole in all this:) And another be represented, as, An obscure person (not known at Court,) that hath a knack perhaps to decipher a letter, or so, (& may be made use of for such a purpose,) but makes no figure in the world, nor looks for any great matters; & may be easyly induced (by a few good words) to do what you would have him: In such case, I say, it is not hard to judge, whether of the two stands fairest to be preferred.

Nor is it sayd, as if I were not willing to serve their Ma^ties^ without being Hired. But the quite contrary. I have been, & shal be willing to serve their Ma^ties^ the best I can upon all occasions. And to do it gratis (if it be their pleasure it should be so:) & do it as honestly as those who are better payd. Of their Ma^ties^ I make no complaint at all. All we can do for them is little inough: & it is impossible they can gratify all.

But when I am imployed (in troublesome work) by persons of Honour: I expect they should do honourable things: And, if therein I am mistaken, it is their fault, not mine. I am

S^r^ your very humble servant

JOHN WALLIS.

Out of the mass of letters on the subject, of which I have copies, a single example may be given in conclusion to show both certain mental characteristics of Wallis and also something of his elementary work with ciphers. The letter was written to James Johnston* whom Swift called "one of the greatest knaves, even in Scotland."

To Mr Johnston Ambassador to Berlin.

Sr June 9, 1692. Soundess by Nettlebed, Oxfordshire.

Your letter to me of May 28. came to Oxford at a time when I was not there; nor have I been there since, but it was sent after me where I now am. I thank you for the kindness you did me, or would have done me, at Berlin. And if your endeavours did not succeed, I am not to blame you for it. Onely I could have wished, you had told ye Elector himself that I had yet received nothing (& he might then use his discretion,) rather than to Monsr Smetteau who knew it well before. Of whom you will give me leave to say, that he might perhaps act like a Statesman, but certainly not like a Gentleman; to treat me like a child, as if I were to be wheedled on to difficult services by a few fair words, & a promise of sugar-plums, which should in the issue signify nothing. I did not want his present, (& perhaps can spare it as well as he,) nor did I ask any; till he told me of his own accord that his Master took my service very kindly, that he intended me an honorable present, & had given him orders to tell me so, & he expected it by ye next Post. Nor did he onely say so to me, but caused it to be spread abroad (for I heard it from divers hands) that ye Present was come, that it was a rich medal with an honorable inscription, & a gold chain of a great value. Now if such present was sent me, but never came to my hand; I was wronged on a double account: first in missing the present that was intended me; & then in being thought Rude by his Electoral Highness for not returning some acknowledgement for the bounty which he thought I had received. But if no such present was sent, nor was intended; He should have sayd nothing of it, & I should have sayd as little. Yet when

* James Johnston (1643?–1737), son of Sir Archibald who was hanged for treason (1663) by order of Charles II.

I had deciphered for them between two & three hundred sheets of very difficult, & very different cipher, they might at lest have afforded me Porters pay; if not that of a Scrivener. Yet I would not be thought in this to reflect on his Electoral Highness, but on Myn-heer Smetteau. For Princes know no more of Things or Persons, in such cases, but as they be represented by their Ministers. His last evasion, of y^e^ Coyners slowness, is so pittyfull an excuse that I am ashamed to hear it. For if a Coyner cannot, in three years, prepare a Medal; a Goldsmith certainly might, in a less time make a Gold-chain; (or somewhat equivalent:) or if he had remitted hither the value in Mony, it might have been made at London. But inough of this.

As to what you desire, concerning an easy Cipher, which yet may be tolerably safe, because you are to use it yourself & cannot well spare much time for it: I think you may venture on such as this,

For y^e^ letters, a b c d e f g h i k l m n o p q r s t u w x y z, put the next after them, b c d e f g h i k l m n o p q r s t u w x y z a, or the next before them, z a b c d e f g h i k l m n o p q r s t u w x y,

or the next but one, or two, before or after them in the Alphabet; and intermix (as Nulls) the numeral figures, 1 2 3 4 5 6 7 8 9 or the greek letters α β γ δ ϵ ζ &c, or what other such notes you please. (And you may by one such alphabet correspond with one person, & by another with another whereof neither shal understand the other's cipher. (Which is a method very ready for use; (And I know that K. James with his correspondents here doth use Ciphers not better than these.)) And 'tis tolerably safe, on these accounts. 1. 'Tis a chance, whether or no the letter be intercepted. 2. If so, 'tis yet a further chance whether those who have it will attempt its being deciphered (& not rather neglect it as thinking it to no purpose.) 3. 'Tis a further chance whether they light upon one who can do it. 4. If deciphered 'tis yet a chance whether it do you any considerable mischief; (for I presume you will not intrust any thing of very great moment therein.) And of all these casualties if any one so happen as you would have them, you are safe. If you desire one which (with a little more trouble) may be

somewhat more safe, I may send you one from Oxford (whither I am now going) from

S^r your humble servant,

These, JOHN WALLIS.

For M^r J. JOHNSTON, Secretary for Scotland;
at my Lord Nottingham's Office, at Whitehall, London.

SHORTER NOTICES.

Analytic Geometry and Calculus. By FREDERICK S. WOODS and FREDERICK H. BAILEY. Ginn and Company, 1917. 516 pp.

THIS book is a revision and abridgment of the authors' Course in Mathematics for Students of Engineering and Applied Science. In making this abridgment the authors have omitted determinants, theory of equations, poles and polars, diameters, center of curvature, special methods of integration, and complex numbers.

The first eight chapters deal entirely with analytic geometry and give the subjects usually given in a first course in American colleges. The following topics are exceptionally well treated: "Variables and functions," "Graphs" and the derivations of the standard equations of the conics. In several places the authors are very careless about a theorem and its converse, i. e., they prove a theorem and then state or use its converse. An example of this is on page 61 where they prove that two perpendicular lines have their slopes negative reciprocals of each other and then conclude that "two lines are perpendicular when the slope of one is the negative reciprocal of the other." Moreover no attention is called to the fact that if the lines are perpendicular to the coordinate axes their slopes are not negative reciprocals.

In Chapter IX they introduce calculus by means of slope and area. This is a very well written introduction to the calculus except for the fact that the definition of limit on page 130 is incorrect (the word "numerically" should be inserted after the word "remains" on line 4). Then follow chapters on the conventional work of maximum and minimum, tangents

and normals, integration, etc. This is followed by Chapter XIII on Applications of integration which is exceptionally well written. The fundamental idea of the summation process of the integral calculus is given in the clearest manner. On page 268, line 10, a lower limit has been omitted and at the bottom of the same page the letter A stands for a point, while on the next page (line 8) it mysteriously becomes an area.

Passing from the best-written chapter in the book we come to the poorest, namely Chapter XIV, on Space geometry. In this chapter the authors have made many inaccurate statements. For example on page 310 we find: "Find the equation of the sphere formed by revolving the circle $x^2 + z^2 = a^2$ about OX as an axis." Now $x^2 + z^2 = a^2$ represents a cylinder and not a circle. The authors evidently meant the circle $x^2 + z^2 = a^2$, $y = 0$. The proof of the distance formula from a plane to a point is also incorrect, for it assumes that the plane cuts the z-axis. The same trouble exists on page 63 in the proof for the distance formula from a line to a point; there it is assumed that the line cuts the y-axis.

The chapter on space geometry is followed by chapters on Partial differentiation, Multiple integrals, Infinite series and a short course in Differential equations.

Many fine examples are worked out in the text and many more are given in the exercises, the total number of which is 2,000. These exercises are to be found only in long lists placed at the end of each chapter. The book is well adapted for use in a course covering both analytics and calculus.

F. M. MORGAN.

Introduction to the Calculus of Variations. By W. E. BYERLY. Cambridge, Harvard University Press, 1917. 8vo. 48 pp. Cloth. Price 75 cents.

THIS little book is the first of a series of "Mathematical Tracts for Physicists." It indicates in admirably clear style the solution of a number of examples involving some of the fundamental ideas of the calculus of variations. As the subject owed its origin to the attempt to solve a rather narrow class of problems in maxima and minima, the eight pages of the Introduction are mainly taken up with a discussion of three simple examples: the shortest line, the curve of quickest descent, the minimum surface of revolution.

The integrals of the Lagrange equations arising in con-

nection with the second and third of these examples are given in the second chapter (pages 9–22) which is entitled: "Variations. Notation and nomenclature. Illustrative problems." We find here also a second solution of the two-dimensional shortest line problem (polar coordinates), and a solution of the problem of the geodesic line joining two given points on the surface of a sphere. Section 10 is devoted to isoperimetrical problems. The seven examples that are given for solution involve slight developments of the text.

In Chapter III (pages 23–28), on "Problems involving several dependent variables," Hamilton's principle and its application are considered. Chapter IV (pages 29–33) on "Multiple integrals" contains (1) the derivation of the differential equation of minimal surfaces, and, by means of Hamilton's principle, (2) the derivation of the differential equation for small transverse oscillations of a stretched elastic string.

"Variation of limits" and the "Principle of least action" are the topics of the last chapter. In each of the last three chapters are examples to be solved.

No references to the literature of the subject are to be found in the tract.

R. C. ARCHIBALD.

Science and the Nation. Essays by Cambridge Graduates with an Introduction by the Right Hon. Lord Moulton. Edited by A. C. SEWARD. Cambridge, at the University Press, 1917. 22 + 328 pages. Price 5 shillings. (G. P. Putnam's Sons, New York. Price $1.50.)

"IT is widely believed that technical education stands for efficiency and prosperity, but pure science is regarded as something apart—a purely academic subject. It was with a view to demonstrate the fallacy of this distinction that the present volume was suggested," writes Mr. Seward.

The volume contains thirteen chapters each written by a specialist of note. The first four chapters, occupying about one third of the volume, are headed as follows: "The national importance of chemistry" by W. J. Pope; "Physical research and the way of its application" by W. H. Bragg; "The modern science of metals, pure and applied" by W. Rosenhain; "Mathematics in relation to pure and applied science" by E. W. Hobson.

Some of the topics in Professor Hobson's very interesting 27-page chapter are: mathematics and practical life; evolution of mathematical concepts; physics and mathematics; dynamical theory of tides; mathematics and physical science; optics and mathematics; abstract dynamics; geometry; reform of teaching; the teaching of mechanics; university teaching; research.

The volume as a whole is exceedingly readable and the reviewer considers that a perusal of its pages must convince any one that pure science is *not* to be regarded "as something apart."

R. C. ARCHIBALD.

Cours de Mécanique. Vol. II. By LÉON LECORNU. Paris, Gauthier-Villars, 1915. 538 pp.

THIS is the second of the three volumes of a treatise on mechanics for use in l'Ecole Polytechnique. The first volume was reviewed in the BULLETIN for April, 1915. The present volume is devoted to theoretical mechanics with a few immediate applications and it is the intention of the author to treat the mechanics of engineering in the third volume.

The first four parts of the complete course are contained in the first volume. The second volume begins with the fifth part, on general dynamics, which includes the fundamental theorems on the motion of a system of particles, the theory of energy, constrained motion, impact, the equations of Lagrange, and small oscillations.

The dynamics of a rigid body is the subject of the sixth part and includes the following topics: moments of inertia, rotation about a fixed axis, motion parallel to a fixed plane, motion about a fixed point, motion of a heavy homogeneous solid of revolution on a smooth horizontal plane, motion of a solid subject to no restraints, and a brief chapter on an extension to the motion of a deformable system.

The seventh part, on the dynamics of physical solids, is devoted principally to a discussion of friction. The theory of elasticity is given in the eighth part, and the volume is concluded by the ninth part on the mechanics of fluids.

W. R. LONGLEY.

Les Théories statistiques en Thermodynamique. Par H. A. LORENTZ. Leipzig, B. G. Teubner, 1916. 120 pp.

DELIVERED at the Collège de France by Lorentz in 1912,

reported and edited by L. Dunoyer in 1913, and finally printed by the author in 1916 with a mathematical appendix supplementing the text, these lectures may be characterized as a frank discussion of some of the most troublesome points in recent thermodynamical theories. Lorentz never throws dust in the reader's eyes; whatever is doubtful is plainly so labeled; that which is tentative is distinguished from that which is settled; the physical ideas are discussed with a minimum of mathematics, and all extended analytical work is reserved for the appendix, where it can neither befog nor interrupt the even tenor of the work.

The relation of entropy to probability in classical thermodynamics, with applications to monatomic gases and other substances, the effect of the size of molecules, the comparison of two possible definitions of probability (Boltzmann and Einstein), canonical ensembles (Gibbs), and fluctuations in statistical phenomena are treated in the first three lectures. The last topic is of great importance and of recent development, with applications to the emission of α particles, to the scattering of light, to Brownian motions, and to black body radiation (fourth lecture). The initial impulse in this field has been largely due to Einstein. Lorentz's connected exposition will be welcomed. A treatment of Planck's formula and of the theory of quanta, admirably honest, closes the set of five suggestive lectures. There is philosophy as well as physics in the work.

There is no other text which can be recommended as comparable with this for clarifying ideas in a very opaque field, and no great knowledge of mathematics or physics is a necessary preparation for enjoying the book, for sensing the exhilarating boldness of some recent theories, and for discerning possible directions of escape from present perplexities.

E. B. WILSON.

The Theory of Electrons and its Applications to the Phenomena of Light and Radiant Heat. By H. A. LORENTZ. Second Edition, Leipzig, B. G. Teubner, 1916. 343 pp.

THIS second edition of Lorentz's Theory of Electrons is practically identical with the first edition, except for a few changes in the footnotes and in the mathematical appendices or notes, which follow the main text. For the most part these changes are for the purpose of giving a clearer state-

ment of the theory of relativity and of the theory of quanta, subjects which were not as prominent in 1909 as they are at present. Except in these two matters, the theory of electrons has not progressed much in the last half dozen years. I need only refer, therefore, to my review of the first edition, BULLETIN, volume 17, pages 194–200.

E. B. WILSON.

Einführung in die theoretische Physik in zwei Bänden. Von CLEMENS SCHAEFER. Bd. I. *Mechanik materieller Punkte, Mechanik starrer Körper und Mechanik der Kontinua (Elastizität und Hydrodynamik).* Leipzig, Veit, 1914. xii + 925 pp. + 249 fig.

THE author, Dr. Schaefer, is well known for his popular and excellent introduction to Maxwell's theory. It was the success of this work, with its clear indication of Schaefer's ability as a writer, that led his friends to press him to publish an introduction to theoretical physics—a large work intended to cover a course of lectures four times a week for five or six semesters. There are not so many general treatises on theoretical physics of the scope of this one that the author need fear the chance of duplicating existing texts.

The present volume deals with mechanics taken in a broad way, covering the mechanics of a particle, of systems of particles, of rigid bodies, and of continua; that is, of fluids and elastic bodies.

The treatment of mechanics of a particle contains chapters upon kinematics, general principles of dynamics, special cases of motion (particularly oscillatory and including Foucault's experiments), general principles of dynamics of systems of particles and their application to special systems. As might be expected, the material does not differ very greatly in sort or in treatment from that found in treatises on the mechanics of a particle. The introduction of Coriolis's theorem on relative motion does, however, come somewhat earlier than usual. A small amount of vector analysis is used, being developed from time to time as it is needed. Moreover, there is a good and unusual section on free and forced oscillations of finite amplitude, which has interesting applications in a theory of sound, and there is a demonstration of Dirichlet that in a statical system the potential energy is at minimum for stable equilibrium—a fact which is often assumed without demon-

stration. Hamilton's principle and Lagrange's equations are also taken up before the theory of the motion of a system of particles is left.

The mechanics of rigid bodies starts with kinematics, in which there is given a considerable discussion of the geometry of rigid motion. The dynamical part is founded upon d'Alembert's principle, and in treating the momental ellipsoid a certain amount of the theory of the linear vector function is presented. The general theory of the dynamics of rigid bodies concludes with a section on statics and it may be well to point out that the author follows modern procedure in making statics depend upon dynamics. There is a considerable treatment of the theory of the top and of potential theory, including Poisson's equation.

The treatment of continua begins with the theory of infinitesimal displacement, of which the treatment is detailed even to the introduction of the divergence and curl. Dynamics, when applied to continuous media, begins with statics,—a procedure just the reverse of that followed in treating rigid bodies. We may well ask whether it is logical to indulge in this chiastic arrangement, and are we likely to maintain it, or shall we ultimately come to begin the theory of continua with kinetics, or may we revert to the old-fashioned method of putting statics everywhere before kinetics. After treating various cases of elastic equilibrium, the author takes up the motion of an infinite medium (wave motion) and follows it with the oscillations of strings and membranes; and here we find Green's functions and integral equations, which we should not have found, at any rate in the same form, in older texts, although Lord Rayleigh and Oliver Heaviside long ago used methods not wholly dissimilar. In more or less parallel fashion the author then takes up the oscillation of bars and plates.

Finally he goes on to hydromechanics, starting with equilibrium and with wave motion. He follows this with a chapter upon motion with a velocity potential including a treatment of Stokes's theorem. He gives particular attention, as is customary, to two-dimensional motion and develops a few cases of discontinuous motion wherein the theory of functions of a complex variable is used. The fundamental properties of vortex motion are well treated, and finally the theory of friction in liquids, which closes with a few words on turbulence and on Prandtl's investigations.

This rapid sketch of the contents of Schaefer's first volume will indicate that the method followed by the author is for the most part to treat fundamental theorems in detail and we may add with clearness, to give a few of the most important applications of the theorems, but to leave aside anything that would be considered as distinctly special and better suited to technical texts on the various branches of mechanics.

Here in America the work can scarcely be used otherwise than as a book of reference. Few universities, if any, offer a course, or are in a position to offer one, which should stretch an introduction to theoretical physics over five or six semesters with four lectures per week. Those of us who give courses on the subject generally find it better to proceed in a very different way; namely, to give in one course perhaps of three hours per week for a year of thirty weeks, a general introduction to mechanics, heat, light, and electricity, leaving the student at the end of the course in a position to undertake serious courses in the various branches of physics, such as rigid mechanics, hydrodynamics, elasticity, optics, thermodynamics, or electricity and magnetism. For our conditions this is undoubtedly a better arrangement because, after the one introductory year, the student is free to turn his attention intensively to any one of the major branches of physics, whereas on the plan followed by Dr. Schaefer, he would be methodically carried through each of the branches in about three years; thus, so far as I can see, getting in no one of those branches the detailed treatment which he would get of one or two of them under our method.

In teaching theoretical physics, it is comparatively easy to give a set of reasonably satisfactory lectures, and it is equally difficult to gather together as satisfactory a set of exercises for the student to work. Unfortunately, though the lecturer may find much in this book which will aid him in the preparation of his lectures, the teacher will find little or nothing which will give him hints of material suitable for assignment as exercises for his pupils, and somehow or other it seems to be true that without exercises American students, even the most industrious, do not gain the grip upon a subject and the insight into it which they should. English books, to a great extent, and French treatises to an increasing extent, are setting forth exercises for the student to work, and it may be hoped that sooner or later this style of composition

may appear in Germany. From some points of view, placing exercises in a book like Dr. Schaefer's might appear uncalled for, and he can hardly be blamed for omitting them, but it is not difficult to believe that even in Germany students would appreciate their presence.

EDWIN BIDWELL WILSON.

NOTES.

THE second summer meeting of the Mathematical Association of America was held at Western Reserve University and the Case School of Applied Science, Cleveland, Ohio, on Thursday and Friday, September 6–7, 1917, immediately following the summer meeting of the American Mathematical Society. The joint dinner of the two organizations took place on Wednesday evening, and a joint session was held on Thursday morning at which Professor L. P. EISENHART, of Princeton University, delivered by invitation an address on "Darboux's contribution to geometry." The further programme included papers by H. E. HAWKES: "Undergraduate mathematical clubs," with discussion led by R. C. ARCHIBALD and D. A. ROTHROCK; E. R. HEDRICK, retiring presidential address: "The significance of mathematics"; E. B. STOUFFER: "Geometry for juniors and seniors," with discussion led by ARNOLD EMCH and L. W. DOWLING; L. C. PLANT: "The treatment of the applications in college courses in mathematics," with discussion led by W. A. HURWITZ and A. M. KENYON. An informal report on the activities of the national committee on mathematical requirements was presented by the chairman, J. W. YOUNG; this report will be published in an early number of the *American Mathematical Monthly*. Ninety persons, representing fifty-one institutions, attended the various sessions. The Council elected nine individuals and two institutions to membership.

THE American association for the advancement of science will hold its seventieth meeting at Pittsburgh from December 28 to January 2 under the presidency of T. W. RICHARDS, of Harvard University. The retiring presidential address will

be by C. R. VAN HISE, president of the University of Wisconsin. H. N. RUSSELL, of Princeton University, is vice-president and F. R. MOULTON, of the University of Chicago, is secretary of Section A.

THE September number (volume 19, number 1) of the *Annals of Mathematics* contains the following papers: "Minimal surfaces applicable to surfaces of revolution," by J. K. WHITTEMORE; "A reduction of certain differential equations of the first order," by W. D. MACMILLAN; "The function $W = L(\tau)$ defined as the inverse of the function $\tau = W - \log W$," by W. D. MACMILLAN; "Solution of the matrix equation $X^{-1}AX = N$," by H. T. BURGESS; "Linear functionals of N-spreads," by C. A. FISCHER; "Correction," by L. I. HEWES; "Substitution groups and possible arrangements of the players at card tournaments," by G. A. MILLER; "Problems of the calculus of variations invariant under a continuous group," by I. A. BARNETT; "A proof of Petersen's theorem," by H. R. BRAHANA; "Complete existential theory of Bernstein's set of four postulates for Boolean algebras," by J. S. TAYLOR; "Ratio, proportion and measurement in the elements of Euclid," by H. B. FINE.

THE English quarterly *Science Progress* has for the last two years published regularly a set of reviews, with full references to current scientific literature, of recent advances in science. With the first number of the twelfth volume (July, 1917) these reports have been given a more prominent place and increased in number and length. Those which are of the greatest interest to mathematicians are on philosophy, by H. ELLIOT; mathematics, by P. E. B. JOURDAIN; astronomy, by H. S. JONES; physics, by JAMES RICE; physical chemistry, by W. C. M. LEWIS; and crystallography, by ALEXANDER SCOTT. Of these the article on mathematics is the longest, and it is of interest to see accounts of recent work appearing much more promptly than is the case with the *Fortschritte* and the *Revue Semestrielle*. The reports are necessarily very short, but there is a welcome attempt at selection and criticism of important papers which marks a certain advance over the procedure of the older journals. It is especially interesting to observe the increasing part taken by America in the mathematical work of the world.

In *Notes and Queries*, London, for August, 1917, J. H. RIVETT-CARNAC has the following note on "Magic squares in India": "In the annual report for 1916 of the Archaeological superintendent of Hindu and Buddhist remains in northern India it is stated that amid the ruins of a temple at Dudhai, Jhansi, was to be seen a stone on which was engraved a magic square: (1) the sum of each row, each column, and each diagonal is 34; (2) the sum of all the numbers in each subsquare is 34. Such squares probably are well known, but to find one in a Hindu temple assigned to the eleventh century A. D. is unusual, perhaps. The figures are 1 to 16."

THE following is a partial list of university and college teachers of mathematics who have recently entered the military service:

Dr. J. W. ALEXANDER, Princeton University, second lieutenant, N. J. national guard. Professor A. A. BENNETT, University of Texas, captain, coast artillery. Professor A. D. BUTTERFIELD, Worcester Polytechnic Institute, captain, aviation branch of signal corps. Dr. J. F. CONNER, Catholic University of America, first lieutenant, bureau of supplies and accounts, navy. Dr. H. B. CURTIS, Columbia University, second lieutenant, aviation corps. Professor PETER FIELD, University of Michigan, captain, coast artillery. Dr. R. E. GILMAN, Cornell University, captain of artillery. Dr. W. L. HART, Harvard University, second lieutenant, field artillery. Mr. L. S. HILL, Princeton University, ensign, navy. Mr. J. W. HOPKINS, Harvard University, second lieutenant, heavy artillery. Professor S. D. KILLAM, University of Alberta, gunner, Canadian artillery. Dr. E. A. T. KIRCHER, Harvard University, captain, coast artillery. Dr. J. V. MCKELVEY, Cornell University, second lieutenant, national army. Dr. T. A. PIERCE, Harvard University, second lieutenant, national army. Professor A. L. UNDERHILL, University of Minnesota, captain, coast artillery. Professor OSWALD VEBLEN, captain, ordnance, reserve corps. Professor K. P. WILLIAMS, Indiana University, captain, field artillery.

PROFESSOR PAUL PAINLEVÉ, of the University of Paris and recently minister of war, has been made premier of the French Republic.

AT Washington University, associate professor W. H.

ROEVER has been promoted to a full professorship of mathematics and appointed acting head of the department.

PROFESSOR P. P. BOYD, of the University of Kentucky, has been appointed dean of the college of arts and sciences and acting president of the university.

AT Pennsylvania State College, associate professors J. H. TUDOR and H. F. STECKER have been promoted to full professorships of mathematics.

DR. R. A. JOHNSON, of Western Reserve University, has been appointed professor and head of the department of mathematics in Hamline College.

DR. IDA BARNEY, of Smith College, has been appointed professor of mathematics at Lake Erie College, Painesville, Ohio.

PROFESSOR A. B. FRIZELL has resigned his professorship of mathematics in McPherson College.

PROFESSOR G. H. SCOTT, of Yankton College, has resigned to become principal of Benzonia Academy, Benzonia, Mich.

AT the University of Chicago, assistant professor A. C. LUNN has been promoted to an associate professorship of mathematics.

AT Northwestern University, assistant professor E. J. MOULTON has been promoted to an associate professorship of mathematics. Mr. F. L. KERR and Mr. A. D. CAMPBELL have been appointed instructors in mathematics.

DR. W. V. N. GARRETSON, of the University of Michigan, has been appointed associate professor of mathematics in Pennsylvania Military College.

PROFESSOR M. O. TRIPP, of Olivet College, has been appointed to an assistant professorship of mathematics in the University of Maine.

AT the University of Toronto, Dr. SAMUEL BEATTY has been promoted to an assistant professorship of mathematics.

DR. C. C. CRUMP, of Carthage College, has been appointed assistant professor of astronomy at Ohio Wesleyan University.

MR. HERMAN BETZ, of Cornell University, has been appointed instructor in mathematics in the University of Michigan.

DR. W. W. DENTON, of the University of Illinois, has been appointed instructor in mathematics in the Worcester Polytechnic Institute.

PROFESSOR C. A. BARNHARDT, of Carthage College, will be a member of the mathematical faculty of Colorado College during the coming academic year.

DR. W. H. BESANT, fellow of St. John's College, Cambridge University, and university lecturer in mathematics, died June 2, 1917, at the age of eighty-eight years.

PROFESSOR S. B. KELLEHER, of the University of Dublin, died August 18, 1917.

BARON DAIROKU KIKUCHI, formerly professor of mathematics in the University of Tokyo and at one time president of that university, died August 20, 1917. Among other offices held by him was that of minister of education and privy councillor. Baron Kikuchi was one of the first Japanese scholars to be educated in England and to introduce modern mathematics into the universities of Japan. He was well known for his contributions to the history of Japanese mathematics.

MRS. E. B. DAVIS, of the Nautical Almanac Office, Washington, D. C., died in April, 1917. Mrs. Davis had been a member of the American Mathematical Society since 1910.

BOOK CATALOGUES:—Deighton, Bell and Company, Cambridge, England, catalogue 28, exact sciences, 1900 titles.—Dulau and Company, 37 Soho Square, London, catalogue 68, natural and mathematical sciences, 1286 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ABHANDLUNGEN über den mathematischen Unterricht in Deutschland, veranlasst durch die Internationale Mathematische Unterrichtskommission. Herausgegeben von F. Klein, in 5 Bänden. Leipzig, Teubner, 1909–1916. Band I: Die höheren Schulen in Norddeutschland. Geh. M. 16.00, geb. M. 18.00. Band II: Die höheren Schulen in Süd- und Mitteldeutschland. Geh. M. 12.00, geb. M. 14.00. Band III in 3 Teilbänden: Einzelfragen des höheren mathematischen Unterrichts. 1ter Teil, geh. M. 10.00, geb. M. 12.00; 2ter Teil, 1te Abt., geh. M. 10.00, geb. M. 12.00; 2te Abt., geh. M. 12.00, geb. M. 14.00. Band IV in 2 Teilbänden: Die Mathematik an den technischen Schulen. 1ter Teil, geh. M. 14.00, geb. M. 16.00; 2ter Teil, geh. M. 17.00, geb. M. 19.00. Band V in 2 Teilbänden: Der Mathematische Elementarunterricht und die Mathematik an den Lehrerbildungsanstalten. 1ter Teil, geh. M. 13.00, geb. M. 15.00; 2ter Teil, geh. M. 9.00, geb. M. 11.00.

CAREY (F. S.). Infinitesimal calculus. Section 1. London, Longmans, 1917. 8vo. 14 + 144 + 5 pp. 6s.

EREDI (F.). Défaut de tout fondement à la méthode des variations d'Euler et Lagrange: la fonction de maximum et de minimum dans les isopérimètres. Ravenna, impr. Nationale, E. Lavagna et fils, 1917. 8vo. 40 pp.

FILIASI (G.). Appunti di fisica e matematica. (Intorno ad alcune odierne vedute sulla costituzione della materia e sulla interpretazione dei fenomeni naturali; fisica, chimica e metachimica; l'unità delle sensazioni; le matematiche transcendentali e la conoscenza delle cose; gli spazî metageometrici e l'evoluzione degli esseri viventi; la natura e la divinità.) Parte I: Le intuizioni. Vol. 2. Napoli, tip. F. Giannini e figli, 1917. 8vo. 56 + 136 pp. L. 3.00

GUTZMER (A.). Die Tätigkeit des deutschen Unterausschusses. (Berichte und Mitteilungen, veranlasst durch die Internationale Mathematische Unterrichtskommission, herausgegeben von W. Lietzmann, 1te Folge, XII.) Leipzig, Teubner, 1916. 8vo. 31 pp. M. 1.00

KLEIN (F.). See ABHANDLUNGEN.

LAZZERI (G.). Analisi infinitesimale. II: Calcolo integrale. (Biblioteca degli studenti, Volumi 363–365.) 16mo. 6 + 152 pp. L. 1.50

LIETZMANN (W.). See GUTZMER (A.).

MORGAN (F. M.). See YOUNG (J. W.).

NELSON (A. L.). Plane nets with equal invariants. (Diss., Chicago.) Chicago, University of Chicago, 1916. 8vo. 25 pp.

REY PASTOR (J.). Introducción a la matemática superior. Estado actual, metódos y problemas. Madrid, Biblioteca Corona, 1916. 8vo. 202 pp. Pes. 3.50

——. Teoria de la representación conforme. Conferencias donadas el Juny de 1915, redactadas per E. Terradas. (Publicaciones del 'In-

stituto de Ciencies. Colección de cursos de física i matemática dirigida per E. Terradas.) Barcelona, Instituto de Estudios Catalanes, Palacio de la Diputacio, 1917. 8vo. 115 pp. Pes. 3.00

SMITH (D. M.). Jacobi's condition for the problem of Lagrange in the calculus of variations. (Diss., Chicago.) Chicago, University of Chicago, 1916. 4to. 18 pp.

TERRADAS (E.). See REY PASTOR (J.).

WHITEHEAD (A. N.). The organization of thought: educational and scientific. London, Williams and Norgate, 1917. 7 + 228 pp. 6s.

YOUNG (J. W.) and MORGAN (F. M.). Elementary mathematical analysis. New York, Macmillan, 1917. 8vo. 12 + 548 pp. $2.60

II. ELEMENTARY MATHEMATICS.

ATWELL (R. K.). First book in general mathematics. New York, P. P. Simmons Company, 1917. 12mo. 12 + 257 pp. $1.25

CAMPBELL (W. A.) and HUGHES (T. A.). Arithmetic by grades; third and fourth years. 2 volumes. New York, Hinds, Hayden and Eldridge, 1917. 12mo. 149 + 170 pp. $0.40 + 0.40

CLARK (L. H.). Methods and problems in percentage and proportion. River Falls, Wis., L. H. Clark, 1916. 38 pp. Paper. $0.12

HEILMAN (J. D.) and SHULTIS (F. W.). Study in addition. (Bulletin, ser. 16, No. 3.) Greely, Colo., State Teachers' College, 1916. 8vo. 16 pp.

HUGHES (T. A.). See CAMPBELL (W. A.).

JONES (B. H.). Number games. Portland, Ore., Westcoast Educators' Association, 1916. 32 pp. Paper. $0.15

KERST (B.). Methoden zur Lösung geometrischer Aufgaben. (Mathematische Bibliothek, Nr. 26.) Leipzig, Teubner, 1916. 12mo. 48 pp. Paper. M. 0.80

LIETZMANN (W.). Riesen und Zwerge im Zahlenreich. (Mathematische Bibliothek, Nr. 25.) Leipzig, Teubner, 1916. 12mo. 56 pp. Paper. M. 0.80

MILLIS (J. F.). See STONE (J. C.).

MILNE (W. J.). Key to Milne's second course in algebra. New York, American Book Company, 1916. 358 pp. $0.88

NEW YORK STATE EDUCATIONAL DEPARTMENT. Syllabus of high school subjects. Mathematics. Albany, New York Education Company, 1916. 8vo. 27 pp. $0.10

PHILIP (M.). Algebra through quadratics. New York, Hinds, Hayden and Eldridge, 1916. 201 pp. $0.92

PRICE (I.). Questions in arithmetic; for drill, test and review. (Progressive Series.) New York, Hinds, Hayden and Eldridge, 1917. 12mo. 103 pp. Paper. $0.45

RANSOM (W. R.). Analysis and computation; an introduction to college mathematics. Medford, Mass., Tufts College, 1916. 12mo. 232 pp. $1.30

SHULTIS (F. W.). See HEILMAN (J. D.).

STONE (C. W.). Standard reasoning tests in arithmetic and how to utilize them. New York, Teachers College, Columbia University, 1916. 8vo. 24 pp. Cloth. $0.80. Paper. $0.55.

STONE (J. C.) and MILLIS (J. F.). Higher arithmetic. Boston, Sanborn, 1917. 12mo. 314 pp. $1.00

III. APPLIED MATHEMATICS.

ATTI del comitato nazionale scientifico-tecnico per lo sviluppo e l'incremento dell'industria italiana. Varese, Arti grafiche varesine, 1917. 4to. 20 pp.

BRIDGEPORT PUBLIC LIBRARY. Engineering mathematics. Bridgeport, Conn., 1916. 4 pp.

BURGATTI (P.). Lezioni di meccanica razionale. Bologna, Zanichelli, 1916. Gr. 8vo. 7 + 501 pp.

COLOMBO (G.). Manuale dell'ingegnere civile ed industriale. 36a e 37a edizione. Milano, Hoepli, 1917. 24mo. 15 + 483 pp.

CORBO (R.). Tavole per determinare con facilità e rapidità gli elementi della retta di altezza (Marcq Saint-Hilaire), non che l'ora vera di bordo, la latitudine con un'altezza circummeridiana, la rotta iniziale e distanza ortodomica fra due punti noti, completate da alcune tavole utili per la navigazione costiera. 2a edizione, riveduta e corretta. Napoli, Scuola tip. Sordomuti, 1916. 4to. 42 pp. L. 4.50

DENIKER (J.). Bibliographie des travaux scientifiques (sciences mathématiques, physiques et naturelles) publiés par les sociétés savantes de la France, dressée sous les auspices du Ministère de l'Instruction publique. Tome I, 3e livraison. Paris, Imprimerie nationale, 1916. 4to. Pp. 401–607.

DÉSORTIAUX (E.). La réforme rationnelle de l'heure. Son importance au point de vue économique et sociale. Paris, Gauthier-Villars, 1917. 15 pp. Fr. 1.00

FERNANDEZ NAVARRO (L.). Cristallografia fisica elemental. Madrid, Suarez, 1917. 8vo. 329 pp. Pes. 10.00

FERRARIS (A.). Nuove tavole prontuarie, ad uso dei tornitori, meccanici, per costrurre viti di passo inglese e decimale senza bisogno di calcolo da parte dell'operaio. 7a edizione, con importantissime aggiunte. Torino, tip. P. Marietti, 1917. 16mo. 12 + 138 pp. L. 3.00

FRATCHER (W. F.). Fratcher's instantaneous 8 hour wage calculator. Detroit, Mich., Instantaneous Calculator Company, 1916. 2 mounted tables, $24\frac{1}{2} \times 40\frac{1}{2}$ cm. $1.00

GIESEKE (F. E.) and MITCHELL (A.). Descriptive geometry. Austin, Tex., Von Boeckmann-Jones Company, 1916. 12mo. 119 pp. $0.75

GIRNDT (M.). Die deutschen bautechnischen Fachschulen und der mathematische Unterricht. (Abhandlungen über den mathematischen Unterricht in Deutschland veranlasst durch die Internationale Mathematische Unterrichtskommission, Band IV, Heft 3.) Leipzig, Teubner, 1916. Gr. 8vo. 6 + 232 pp. M. 7.20

LEVI (C.). Trattato teorico-pratico di costruzioni civili, rurali, stradali ed idrauliche. Vol. 1. 4a edizione riveduta. (Biblioteca tecnica.) Milano, Hoepli, 1917. 8vo. 15 + 747 pp. L. 15.00

MARCANTONIO (I.). Questioni idrauliche nella regioni collinari dell' Italia centrali e meridionale. Lanciano, tip. Masciangelo, 1917. 8vo. 23 pp.

MITCHELL (A.). See GIESECKE (F. E.).

MÜLLER (E.). Lehrbuch der darstellenden Geometrie für technische Hochschulen. 2ter Band. Leipzig, Teubner, 1916. Gr. 8vo. 10 + 361 pp. Geh. M. 12.80. Geb. M. 14.00.

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